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CAMBRIDGE INTERNATIONAL PHYSICS NOTES β RELEASE

FOR MY STUDENTS

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Contents

1 *Physical Quantities*

Units of measurement

Any physical quantity contains a numerical value and its associated unit. The system of units of measurement used throughout the scientific world is the **SI units**[1]

[1] SI units, abbreviated from the French *Système Internationale d'Unités*, means the International System of Units. Those who are interested in the history and evolution of the SI can check out the Wikipedia article: `https://en.wikipedia.org/wiki/International_System_of_Units`

SI base units

SI defines seven units of measure as a basic set, known as the **SI base units**

base quantity	base unit	symbol
mass	kilogram	kg
length	metre	m
time	second	s
electric current	ampere	A
temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

Derived units

The seven[2] SI base units are building blocks of the SI system, all other quantities are derived from the base units.

[2] Luminous intensity is beyond the scope of the A-Level syllabus. You are only required to know the other six SI base quantities and their units.

Example 1.1 Give the SI base units of (a) speed, (b) acceleration, (c) force, (d) work done.

$$\text{speed} = \frac{\text{distance}}{\text{time}} \Rightarrow [v] = \frac{[s]}{[t]} = \frac{\text{m}}{\text{s}} = \text{m s}^{-1}$$

$$\text{acceleration} = \frac{\text{speed}}{\text{time}} \Rightarrow [a] = \frac{[v]}{[t]} = \frac{\text{m s}^{-1}}{\text{s}} = \text{m s}^{-2}$$

$$\text{force} = \text{mass} \times \text{acceleration} \Rightarrow [F] = [m][a] = \text{kg m s}^{-2}5$$

$$\text{work} = \text{force} \times \text{distance} \Rightarrow [W] = [F][s] = \text{kg m s}^{-2} \times \text{m} = \text{kg m}^2\text{s}^{-2}$$

Metric prefixes

To save writing so many figures, prefixes are used to indicate multiples and sub-multiples of original units

name	symbol	meaning	name	symbol	meaning
pico	p	10^{-12}	hecto	h	10^2
nano	n	10^{-9}	kilo	k	10^3
micro	μ	10^{-6}	mega	M	10^6
milli	m	10^{-3}	giga	G	10^9
centi	c	10^{-2}	tera	T	10^{12}
deci	d	10^{-1}			

Example 1.2 Alternative units of measurement for length

The radius of the Earth is about 6,370 km.
The width of a human hair is around 60 ~ 90 μm.
The diameter of a water molecule is about 0.3 nm.
The atomic radius of oxygen is about 60 pm.

Dimensional analysis

If an equation is correct, then the units on both sides must be the same. An equation with consistent units is said to be **homogeneous**.

Dimensional analysis is widely used as a rough guide to check for the correctness of equations. There are times when the dependence of one physical quantity on various other quantities cannot not be seen easily, but it might give us helpful hints by merely investigating their units.

There are *unit free*, or *dimensionless* quantities that do not have units. Examples of these are real numbers (2, $\frac{4}{3}$, π, etc.), coefficient of friction (μ), refractive index (n), etc. [3]

Example 1.3 A ball falls in vacuum, all its gravitational potential energy converts into kinetic energy. This is expressed by the equation: $mgh = \frac{1}{2}mv^2$. Show that this equation is homogeneous.

LHS: $[mgh] = [m][g][h] = \text{kg} \times \text{ m s}^{-2} \times \text{m} = \text{kg m}^2\text{s}^{-2}$
RHS: $\left[\frac{1}{2}mv^2\right] = [m][v]^2 = \text{kg} \times (\text{ m s}^{-1})^2 = \text{kg m}^2\text{s}^{-2}$
so we see the equation $mgh = \frac{1}{2}mv^2$ is homogeneous

Example 1.4 The speed of a wave travelling along an elastic string is determined by three things: the tension T in the string, the length L of the string, and the mass m of the string. Let's assume $v = T^a L^b m^c$, where a, b, c are some numerical constants. Find the values of a, b and c.

RHS: $[T]^a[L]^b[m]^c = (\text{kg m s}^{-2})^a \text{m}^b \text{kg}^c = \text{kg}^{a+c}\text{m}^{a+b}\text{s}^{-2a}$
for the equation to be homogeneous, we must have:

$$\text{kg}^{a+c}\text{m}^{a+b}\text{s}^{-2a} = \text{m s}^{-1} \quad \Rightarrow \quad \begin{cases} \text{kg}: & a+c=0 \\ \text{m}: & a+b=1 \\ \text{s}: & -2a=-1 \end{cases} \quad \Rightarrow \quad \begin{cases} a=\frac{1}{2} \\ b=\frac{1}{2} \\ c=-\frac{1}{2} \end{cases}$$

so wave speed is given by: $v = T^{1/2}L^{1/2}m^{-1/2}$, or $v = \sqrt{\frac{TL}{m}}$
this happens to be the correct formula for the wave speed on a string

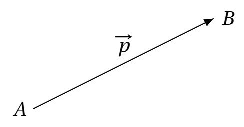
Scalars & Vectors

Physical quantities come in two types: scalars and vectors.

a **scalar** quantity has magnitude only
a **vector** quantity has magnitude and direction

In other words, a scalar can be described by a single number. Examples of scalars are time, distance, speed, mass, temperature, energy, density, volume, etc.

A vector, on the other hand, is usually represented by an arrow in a specific direction. A vector $\vec{p}$ pointing from A to B is shown. The length of the arrow shows the magnitude of the vector, the direction of the arrow gives the direction of the vector. Examples of vectors are displacement, velocity, acceleration, force, field strength, etc.

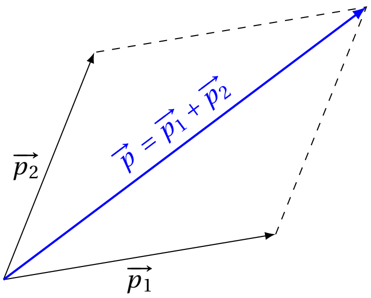


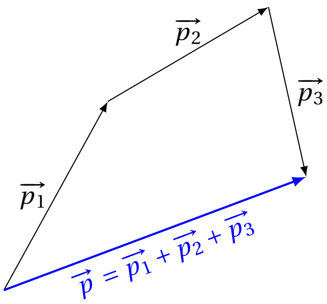
Adding Vectors

Scalar algebra is just ordinary algebra - one can add and subtract scalar quantities in the same way as if they were ordinary numbers, for ex-

[3] Note a correct equation must be homogeneous, but the converse may not be true. Possible problems include an incorrect coefficient, extra term, an incorrect sign, etc.

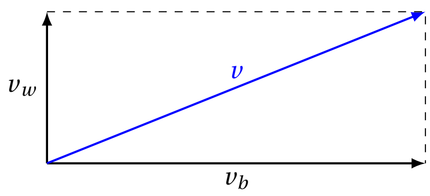
ample, a set of objects with mass $m_1, m_2, \cdots, m_n$ have a total mass of $M = m_1 + m_2 + \cdots + m_n$
Vector algebra on the other hand is more complicated, since we need keep track of the direction
Vectors can be added to form a **resultant** vector - to deal with vector sums, we need take the directions of vectors into account.
Let's consider the sum of two vectors $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$:
The resultant vector $\overrightarrow{p} = \overrightarrow{p_1} + \overrightarrow{p_2}$ lies on diagonal of the parallelogram subtended by $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$, this is called the **parallelogram rule** for vector addition. If the resultant of several vectors $\overrightarrow{p} = \overrightarrow{p_1} + \overrightarrow{p_2} + \cdots \overrightarrow{p_n}$ is to be found, one can join these vectors head-to-tail, the resultant is given by the arrow connecting the tail of $\overrightarrow{p_1}$ to the head of $\overrightarrow{p_n}$:



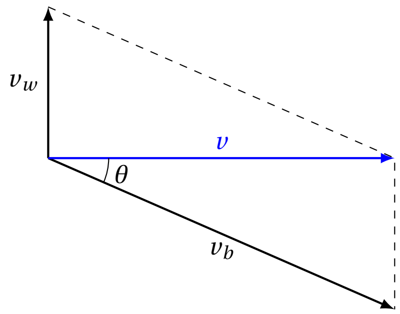


Example 1.5 A river flows from south to north with a speed of 2.0 m s^{-1} and the speed of a boat with respect to the water flow is 5.0 m s^{-1}. (a) Suppose the boat leaves the west bank heading due east, what is the resultant velocity of the boat? (b) If the boat is to reach the exact opposite bank across the river, what is the resultant velocity and in what direction should the boat be headed?

Vector diagrams for the resultant velocity of the boat are illustrated below for part (a) boat heading due south:
magnitude of resultant velocity: $v = \sqrt{v_b^2 + v_w^2} = \sqrt{5.0^2 + 2.0^2} \approx$ 5.4 m s^{-1}.
In this case, the boat reaches opposite bank in shortest time but will drift downstream.



For part (b), the boat is to hit the opposite bank: magnitude of resultant velocity: $v = \sqrt{v_b^2 + v_w^2} = \sqrt{5.0^2 - 2.0^2} \approx 4.6\ \mathrm{m\ s^{-1}}$. in this case, the boat reaches opposite bank in shortest distance but the boat is headed slightly upstream: $\theta = \sin^{-1}\frac{v_w}{v_b} = \sin^{-1}\frac{2.0}{5.0} \approx 24°$

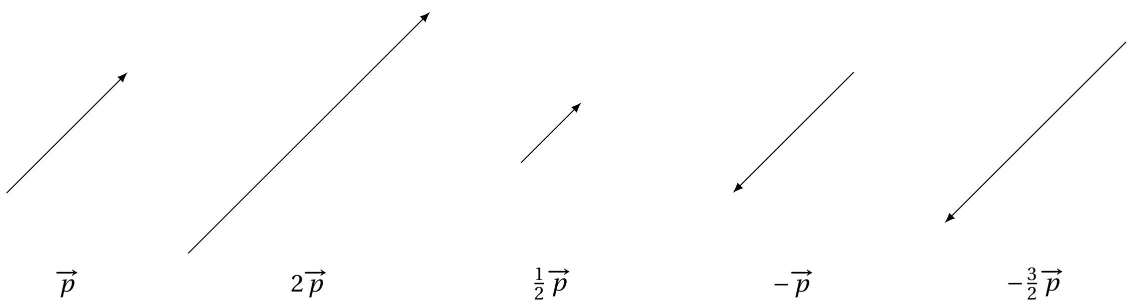


Multiplication of vectors

Vectors can be multiplied by scalars easily, [4] when being multiplied by a scalar number, magnitude of the vector changes. If this number is positive, the vector becomes longer or shorter, but still points in same direction, if the number to be multiplied is negative, the operation reverses the vector's direction...

Example 1.6 Given a vector $\overrightarrow{p}$, the graphical representations of $2\overrightarrow{p}$, $\frac{1}{2}\overrightarrow{p}$, $-\overrightarrow{p}$, $-\frac{3}{2}\overrightarrow{p}$ are:

[4] It is also possible to multiply vectors with vectors, but we'll not go this far in A-level Physics. There are basically two ways of doing vector multiplication: the *dot product* and the *cross product*. Both vector products are useful in physics, but we will not go into the details. You will learn more about vector multiplication in the A-Level mathematics course.⚠

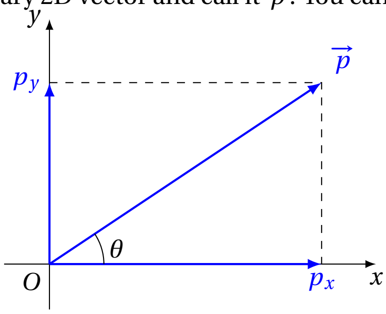


Resolving vectors

Just as it's useful to combine two vectors into a resultant, one can also resolve a single vector into two (or more) components. [5]

[5] This depends on the number of dimensions of space we are working with.

Lets draw an arbitrary 2D vector and call it $\overrightarrow{p}$: You can hopefully see here



how the vector $\overrightarrow{p}$ can be split into two perpendicular components:

– a horizontal component p_x

– a vertical component p_y

Given that $\overrightarrow{p}$ forms an angle θ to the x-axis, then:

$$p_x = p\cos\theta$$

$$p_y = p\sin\theta$$

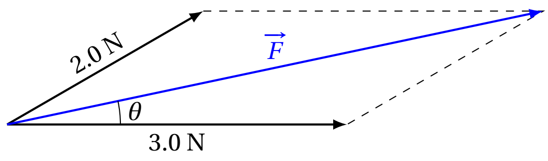
$$p = |\overrightarrow{p}| = \sqrt{p_x^2 + p_y^2}$$

,

$$\tan\theta = \frac{p_y}{p_x}$$

Example 1.7 A force of 3.0 N towards east and a force of 2.0 N towards 30° north of east act on an object. Find the magnitude and the direction of the resultant force.

suppose an arrow of length 1 cm represents a force of 1 N
one can draw a *scale diagram* with a ruler and a protractor as shown



one can find length of the resultant vector is about 4.8 cm
also it forms an angle of about 12° to the 3.0 N force
so resultant force is of 4.8 N acting towards 12° north of east
alternatively, one can find components of the resultant as the sum of individual components

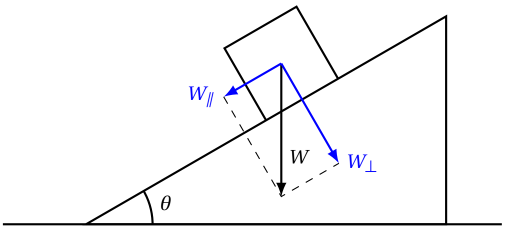
$$F_x = 3.0 + 2.0\cos 30° \approx 4.73\text{ N}, \quad F_y = 2.0\sin 30° = 1.0\text{ N}$$

magnitude and direction of the resultant can then be found from its components

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4.73^2 + 1.0^2} \approx 4.84\text{ N}, \theta = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{1.0}{4.73} \approx 11.9°$$

this of course agrees with scale diagram method, but resolving gives more precise results

Example 1.8 A box of weight $W = 20.0$ N is resting on an inclined slope at 30° to the horizontal. Find the components of weight parallel to the slope and normal to the slope.



the vector diagram is shown
component of weight parallel to slope: $W_{\parallel} = W\sin\theta = 20.0 \times \sin 30° = 10.0$ N
component of weight normal to slope: $W_{\perp} = W\cos\theta = 20.0 \times \cos 30° \approx 17.3$ N

Questions

SI units

Question 1.1 What are the SI base units of (a) density, (b) pressure, (c) energy, (d) electric charge?

Question 1.2 For a substance of mass m, the heat energy Q needed to change its temperature by ΔT is given by: $Q = cm\Delta T$. Find the SI base units of the constant c.

Dimensional analysis

Question 1.3 The resistive force F on a metal ball falling at low speeds in water is given by the equation $F = krv$, where r is the radius of the metal ball, v is its speed and k is a constant. Find the base units of k in the SI system.

Question 1.4 The speed of sound in air can be given by $c = \sqrt{\frac{\gamma p}{\rho}}$, where p is the pressure of the air and ρ is the air density. Show that γ is unit free.

Question 1.5 The effective power output from a wind turbine is given by the equation $P = \frac{1}{2}\eta\rho A v^n$, where ρ is the air density, A is the area of the turbine blades, and v is the wind speed. Given that η is a constant with no units, what is the value of n?

Vector algebra

Question 1.6 An aircraft, which has a speed of 35 m s^{-1} in still air, is flying from south to north at a speed of 32 m s^{-1} with respect to a stationary observer on the ground. Find the magnitude and the possible directions of wind velocity.

Question 1.7 Three forces of 5.0 N, 12 N and 13 N act at one point on an object. The angles at which the forces act can vary. What is the maximum and the minimum resultant force?

2 Kinematics

Kinematics is the study of motion. We will introduce three kinematic quantities, displacement, velocity and acceleration, and see how these terms are used to describe an object's motion.

Kinematic quantities

Displacement & Distance

In everyday language, we talk about the **distance** travelled by an object, which usually refers to the length travelled by an object without considering in which direction it moves, to fully describe position of an object however, we also need specify where it is heading.

> **displacement** is defined as the distance moved by an object in a specific direction

Displacement is the *straight-line* distance pointing from starting point towards end point -even if actual path taken is curved, ie it is a vector quantity, while distance is a scalar. Displacement is always the straight-line distance. Displacement and distance are measured in metres, or any reasonable length units.

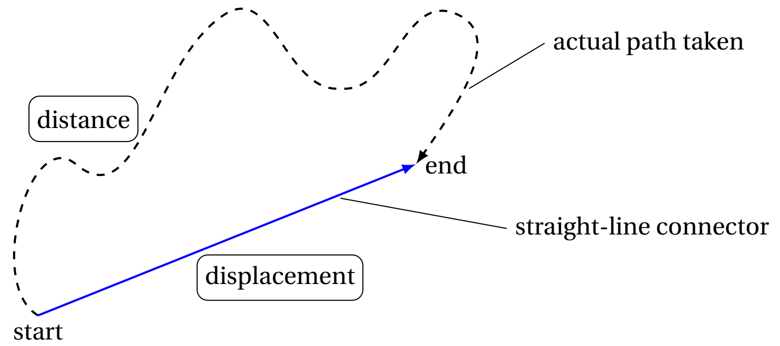
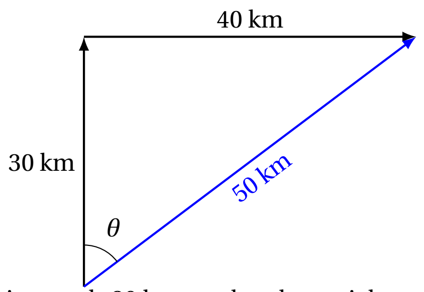


Figure 2.1: difference between displacement and distance

Example 2.1 An athlete is running around a circular track of radius 60 m. When he completes one lap, what is the distance moved out? What about his displacement?

Distance moved is the perimeter of the circle: $s = 2\pi r = 2\pi \times 60 \approx 380$ m
athlete returns to same starting point after one lap, so displacement is zero



Example 2.2 A ship travels 30 km north, takes a right, and then travels 40 km east to reach its destination. Compare the distance and the displacement travelled.

Summing of all lengths gives distance: $30 + 40 = 70$ km
The displacement vector is shown on the diagram.
The magnitude of displacement $= \sqrt{30^2 + 40^2} = 50$ km
at an angle of $\theta = \tan^{-1} \frac{40}{30} \approx 53°$ east of north

Velocity & speed

The displacement of a moving object may change with respect to time, the rate of this change in displacement gives us the velocity with which it's moving.

The change in displacement per unit time is called the **velocity** of the object velocity:

$$v = \Delta s / \Delta t$$

Velocity is a vector quantity which follows from the fact that displacement is a vector quantity. The SI unit of measurement for velocity: $[v] =$ m s^{-1}
In the case of vector quantities in one dimension, *linear* motion, we always assign a direction and indicate this with positive and negative signs. A negative velocity implies motion in the opposite direction to positive.
Defining equation $v = \frac{\Delta s}{\Delta t}$ gives the *average* value for velocity or speed over an interval Δt more precisely:

$$\text{average velocity} = \frac{\text{total displacement}}{\text{time taken}}$$

and...

$$\text{average speed} = \frac{\text{total distance}}{\text{time taken}}$$

This should be distinguished from the notion of *instantaneous velocity*instantaneous velocity; defined as the rate of change in displacement at a particular instant.
If we take a very short interval Δt, as $\Delta t \to 0$, average velocity tends to instantaneous velocity, this is expressed in a compact differential form: $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \Rightarrow$

$$v = \frac{ds}{dt}$$

It is also common to use **speed** to describe how fast an object moves, the speed is defined as the change of the distance travelled per unit time. Velocity can be thought as speed in a particular direction.
Example 2.3 A cyclist travels a distance of 3.0 km in 20 minutes. She rests for 15 minutes. She then covers a further distance of 5.1 km in a time of 40 minutes. Calculate the average speed of the cyclist in m s^{-1}: (a) during the first 20 minutes of the journey, (b) for the whole journey.

For the first 20 minutes: $v = \frac{3.0\times10^3}{20\times60} = 2.5\ \text{m s}^{-1}$

For whole journey: $v = \frac{(3.0+0+5.1)\times10^3}{(20+15+40)\times60} = 1.8\ \text{m s}^{-1}$

Example 2.4 A man walks along a straight road for a distance of 800 m in 5.0 minutes. He then turns around, and walks along the same road for a distance of 280 m in 3.0 minutes. What is the average speed and the average velocity of this man during the 8.0 minutes?

Total distance travelled = 800 + 280 = 1080m, so average speed: $v = \frac{1080}{8.0\times60} = 2.25\ \text{m s}^{-1}$

change of displacement = 800 + (−280) = 520m, so average velocity: $v = \frac{520}{8.0\times60} \approx 1.08\ \text{m s}^{-1}$

Example 2.5 A maglev train travels at an average speed of 60 m s^{-1} from the city centre to the airport, and at 40 m s^{-1} on its return journey over the same distance. What is the average speed of the round-trip? What about the average velocity?

Suppose the distance between airport and city centre is S.
Average speed: $v = \frac{2S}{t_1+t_2} = \frac{2S}{\frac{S}{60}+\frac{S}{40}} = 48\ \text{m s}^{-1}$
For a round-trip, train returns to same staring position so change in displacement is zero. Average velocity is therefore *Zero*

Acceleration

The velocity of a moving object may change, i.e., objects can speed up, slow down **or make turns**

Change in velocity per unit time is defined as the **acceleration**:

$$a = \frac{\Delta v}{\Delta t}$$

– unit of measurement for acceleration: $[a] = \text{ m s}^{-2}$

Acceleration[1] is a vector quantity, it has both magnitude and direction. Because of vector nature of velocity, change in velocity must also have direction. When we're considering *linear* motion, we usually pick direction of initial velocity as positive direction, often left to right.

– $a > 0$ would imply acceleration in the normal sense, i.e., motion with an increasing speed.

– $a < 0$ would imply deceleration, i.e., motion with a decreasing speed.

When velocity changes, it could be change in magnitude or/and change in direction[2] for example, for an object moving along a curved path, its velocity is constantly changing direction, so it must have a non-zero acceleration. An acceleration of Zero on the other hand would imply no change in speed and no change in direction of motion.

The defining equation, which gives average acceleration over time interval Δt is:

$$a = \frac{\Delta v}{\Delta t}$$

We can likewise introduce *instantaneous acceleration* as the rate of change in velocity. Taking the limit where the time interval $\Delta t \to 0$. we have: $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \Rightarrow$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

[1] Joe often writes this as "$Accl^n$" on the board. Useful shorthand that!

[2] Acceleration of an object can be considered as the combination of two components. One component is known as the *normal* acceleration or the *centripetal* acceleration, which is at right angle to the velocity and is responsible for the change in direction of motion. The other component is called the *tangential* acceleration, which is parallel to the direction of motion and causes change in magnitude of object's velocity. You will learn more about these in further mechanics.

Example 2.6 A ball hits a barrier at right angles with a speed of 15 m s^{-1}. It makes contact with the barrier for 30 ms and then rebounds with a speed of 12 m s^{-1}. What is the average acceleration during the time of contact?

note that direction of velocity changed during rebound, so

$\Delta v = 15 - (-12) = 27 \text{ m s}^{-1}$

average acceleration: $a = \frac{\Delta v}{\Delta t} = \frac{27}{30 \times 10^{-3}} = 900 \text{ m s}^{-2}$

Motion graphs

How one quantity changes with another quantity can be visually shown on a *graph*, changes in displacement, velocity or acceleration over time can be shown on *motion graphs*. As we will see, s-t graph, v-t graph and a-t graphs are closely interrelated to one another.

Displacement-time graphs

A displacement-time graph shows an object's position at any given time:

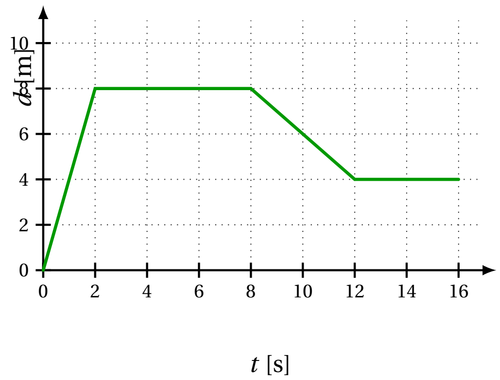


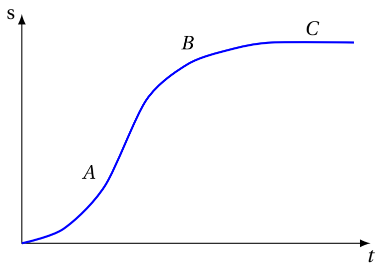
Figure 2.2: A generic displacement-time graph

The gradient of the graph (or the tangent) gives rate of change in displacement, this is instantaneous velocity, which can be given by $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, so we have:

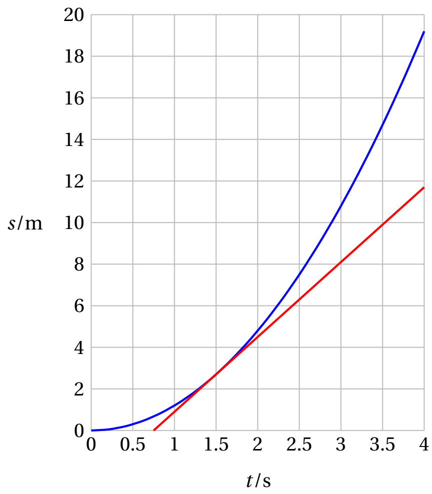
$$\text{velocity} = \text{gradient of } s\text{-}t \text{ graph}$$

Example 2.7 Describing the motion from the displacement-time graph shown.

stage A: gradient of the graph is increasing, showing that the object is speeding up
stage B: gradient starts to decrease, so the object gradually slows down
stage C: curve becomes horizontal, gradient becomes zero, means that the object eventually comes to a stop



Example 2.8 The diagram shows the displacement-time graph for a vehicle travelling along a straight road. Use the graph to find, (a) the average velocity during the first 4.0 s of the motion, (b) the velocity of the vehicle *at* time $t = 1.5$ s.



during first 4.0 s, average velocity is

$$v = \frac{\Delta s}{\Delta t} = \frac{19.2}{4.0} \approx 4.8 \text{ m s}^{-1}$$

to find velocity at $t = 1.5$ s, a tangent is drawn
gradient of tangent gives instantaneous velocity:

$$v = \frac{11.6-0}{4.0-0.75} \approx 3.6 \text{ m s}^{-1}$$

Velocity-time graphs

A velocity-time graph shows the velocity of a moving object at any instant, since the rate of change of velocity gives the acceleration;

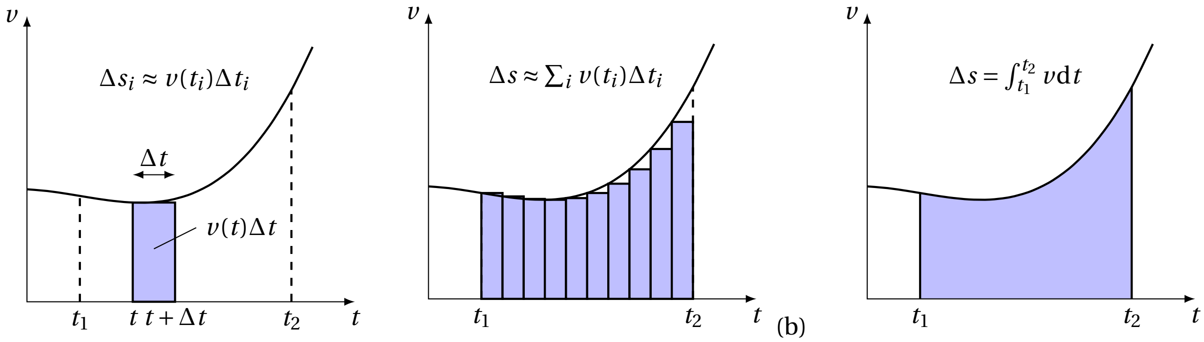
acceleration = gradient of v-t graph

A v-t graph also gives information about the change in displacement, that is:

change in displacement = area under v-t graph

The reasoning here is that in very short interval Δt_i, change in velocity is small so $v(t_i) \approx$ constant during Δt_i. Following the logic, the displacement changed by $\Delta s_i \approx v(t_i)\Delta t_i$, which corresponds to area of a thin rectangle. The sum of all these small Δs_i's gives total change in displacement over a period of time. If we now consider the limit where each of the time interval $\Delta t_i \to 0$ then the total area of these rectangles approximates area bounded by the v-t curve and time axis. [3]

[3] Mathematically, integration is the inverse operation of taking derivatives. By definition $v = \frac{ds}{dt}$, then it follows naturally that $\Delta s = \int v\,dt$. While the derivative of a given function gives the gradient of tangent at each point on its graph, integrating a function gives the signed area bounded by the graph. The reader may find the formal treatment of this relationship in any calculus textbook.



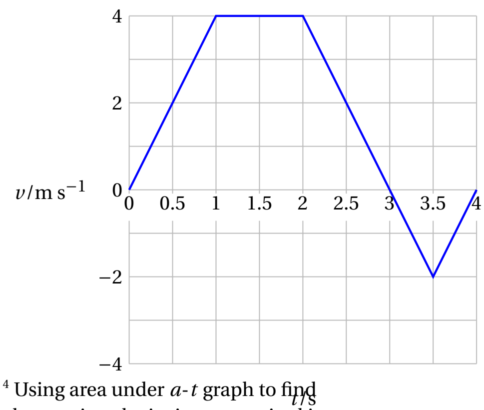
(a) displacement Δs in short interval Δt

(b) total displacement estimated by summing the many Δs_i's

(c) total displacement as area under v-t graph

Figure 2.3: calculating change in displacement by finding the area under velocity-time graph

Example 2.9 The velocity of a toy car is shown. For the journey shown on the graph, use the graph to find (a) the total distance travelled, and (b) the total displacement travelled.



Distance is estimated using area under v-t graph

0~3.0 s: $s_1 = \frac{1}{2} \times 1.0 + 3.0 \times 4.0 = 8.0$ m

3.0~4.0 s: $s_2 = \frac{1}{2} \times 1.0 \times 2.0 = 1.0$ m

total distance $= 8.0 + 1.0 = 9.0$ m

total displacement $= (+8.0) + (-1.0) = 7.0$ m

Acceleration-time graphs

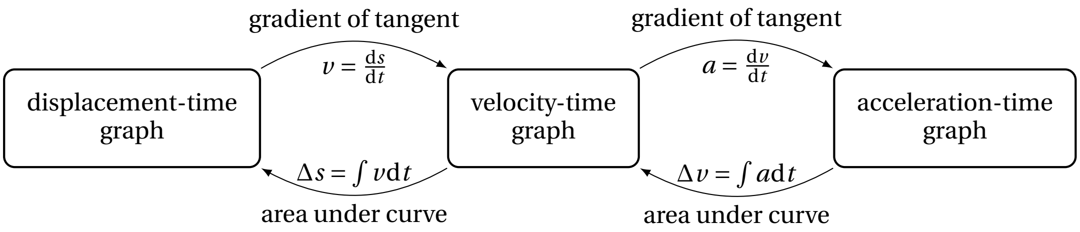
In the same way as above, one can similarly plot an acceleration-time graph to give the changes in acceleration, ie. a-t graphs can give information about changes in velocity. Similar discussions lead to the following conclusion:[4]

change in velocity = area under a-t graph

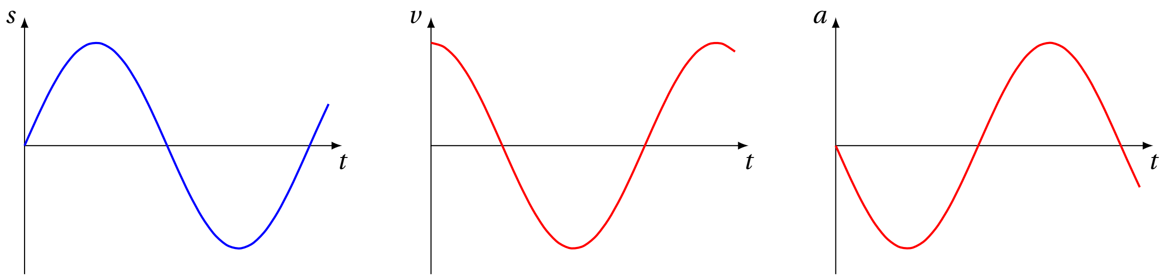
[4] Using area under a-t graph to find changes in velocity is not required in the AS-Level physics syllabus. I am putting this in the notes mainly for the completeness of the discussions on motion graphs.

You'll meet these again [5] in Mechanicslater on, however the relationships between displacement, velocity and acceleration graphs are summarised below:

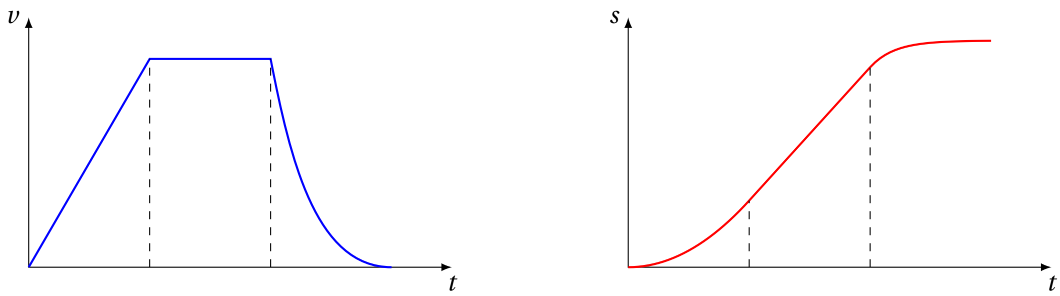
[5] and properly!



Example 2.10 Given the displacement-time graph as shown, check yourself that this s-t graph leads to the velocity-time graph and the acceleration-time graph shown.



Example 2.11 Given the velocity-time graph as shown, check yourself that this v-t graph leads to the displacement-time graph as shown.



Linear motion with constant velocity

let's look at the simplest kind of motion, an object moving at constant speed in a straight line: v = constant.
In this case the equations of motion are the straightforward ones met at GCSE:

$$a = 0$$

and

$$s = vt$$

[6] Where S, represents displacement, V, the velocity and t, the time. You have met these a lot, we'll not say much more about them...

[6] It is implicitly assumed that the motion starts from the origin with respect to which displacement is defined. More generically, we should write: $s = s_0 + vt$, where s_0 is the initial displacement.

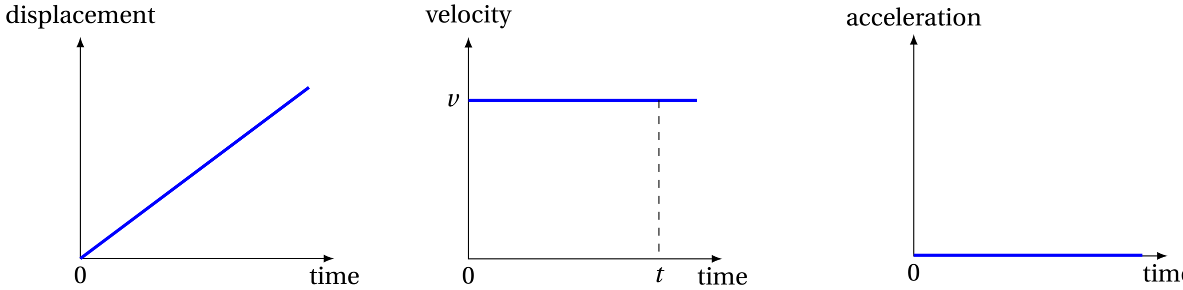


Figure 2.4: Motion graphs for linear motion at constant velocity

Linear motion with constant acceleration

More complex motion will involve a change in velocity and we limit ourselves at A-level to situations where this change in velocity is constant. ie we have linear motion with constant acceleration $a = \text{constant}$. When we do this we see that our motion graphs look as below:

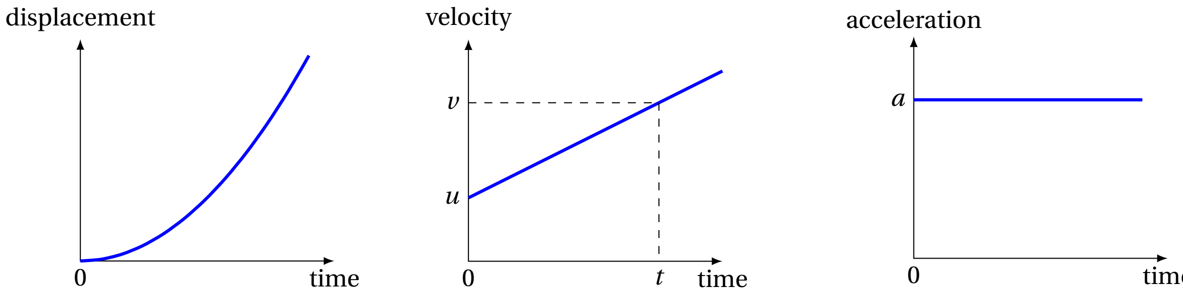


Figure 2.5: motion graphs for linear motion at uniform acceleration

Equations of motion

In order to be able to solve problems relating to motion we need more powerful tools than we have met at GCSE. By combining the equation for acceleration with the definitive equation for velocity, we can come up with a set of motion equations commonly referred to as "the SUVAT equations".
During a time interval t, suppose velocity changes from initial value u to final value v. From the defining equation of acceleration $a = \frac{\Delta v}{\Delta t} = \frac{v-u}{t}$, we get[7]:

$$v = u + at \tag{2.1}$$

This is the first equation in our series.
To find total displacement travelled, we compute the area under the v-t graph

$$s = \frac{1}{2}(u+v)t \tag{2.2}$$

for which we can interpret $\bar{v} = \frac{1}{2}(u+v)$ as the average velocity during that time
plug (2.1) into (2.2), we find an expression for the displacement travelled in terms of u and a:[8] [9]

[7] Proof with calculus: $\mathrm{d}v = a\mathrm{d}t \Rightarrow \Delta v = \int_u^v \mathrm{d}v = \int_0^t a\mathrm{d}t \Rightarrow v - u = at \Rightarrow v = u + at$

[8] Proof with calculus: $\mathrm{d}s = v\mathrm{d}t \Rightarrow \Delta s = \int_0^s \mathrm{d}s = \int_0^t v\mathrm{d}t \Rightarrow s = \int_0^t (u+at)\mathrm{d}t = \left(ut + \frac{1}{2}at^2\right)\Big|_0^t \Rightarrow s = ut + \frac{1}{2}at^2$

[9] Equation (2.3) assumes a zero initial displacement at $t = 0$. If there is a non-zero initial displacement, one should write $s = s_0 + ut + \frac{1}{2}at^2$. Similar discussion applies to equation (2.2).

$$s = ut + \frac{1}{2}at^2 \tag{2.3}$$

This shows the displacement s is a quadratic function in time t, consistent with the parabolic shape of the s-t graph shown. We can also eliminate the time variable t to derive one last equation from (2.1) we have $t = \frac{v-u}{a}$, substitute this into (2.2), we find

$$s = \frac{1}{2}(u + v) \times \frac{v - u}{a} \quad \Rightarrow \tag{2.4}$$

$$2as = v^2 - u^2$$

Example 2.12 A car starts from rest and accelerates uniformly at 5.0 m s^{-2} for 6.0 s.
(a) How fast is the car travelling at $t = 8.0$ s?
(b) What is the distance travelled by the car in this time?

$$v = u + at \quad \Rightarrow \quad v = 0 + 5.0 \times 6.0 = 30 \text{ m s}^{-1}$$
$$s = ut + \frac{1}{2}at^2 \quad \Rightarrow \quad s = 0 + \frac{1}{2} \times 5.0 \times 6.0^2 = 90 \text{ m}$$

Example 2.13 A car is travelling at 30 m s^{-1}. A hazard appears in front of the car, and the driver takes immediate action to stop the car. When brakes are applied, deceleration of the car is 5.0 m s^{-2}. What is the braking distance?

$$2as = v^2 - u^2 \quad \Rightarrow \quad s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 30^2}{2 \times (-5.0)} = 90 \text{ m}$$

Example 2.14 At the instant the traffic light turns green, a motorcycle waiting at the stop line starts with a constant acceleration of 2.0 m s^{-2}. At the same instant, a truck at a constant speed of 16 m s^{-1} overtakes and passes the motorcycle. How far beyond the stop line will the motorcycle overtake the truck?

Lets suppose overtake occurs at time t after motorcycle starts to accelerate, so the distance travelled by motorcycle:

$$s_m = u_m t + \frac{1}{2}at^2 \quad \Rightarrow \quad s_m = 0 + \frac{1}{2} \times 2.0 \times t^2$$

In the same time the distance travelled by truck:

$$s_t = v_t t \quad \Rightarrow \quad s_t = 16t$$

The overtake when:

$$s_m = s_t \quad \Rightarrow \quad \frac{1}{2} \times 2.0 \times t^2 = 16t \quad \Rightarrow \quad t = 16 \text{ s}$$

Substitute t into either s_m or s_t, one finds distance travelled: $s = 256$ m

Free fall

A typical example of uniformly accelerated motion is the free fall everything has the tendency to fall towards ground due to earth's gravity, in this section, we assume that effects of air resistance are negligible, acceleration of free fall is then constant $a = g$. The value of g does not depend on mass of the falling object[10] Near surface of earth, the value of the acceleration due to gravity is: $g \approx 9.81 \text{ m s}^{-2}$, of course value of g could be different on a different planet[11]

For a freely-falling object released from rest, its velocity increases with time as

$$v = u + at = 0 + gt \quad \Rightarrow \quad v = gt$$

and the distance it has fallen from the point of release is:

$$s = ut + \frac{1}{2}at^2 = 0 \cdot t + \frac{1}{2}gt^2 \quad \Rightarrow \quad s = \frac{1}{2}gt^2$$

Example 2.15 An object is released from rest from a height of $h = 24$ m and falls freely under gravity. Air resistance is negligible.
(a) How long does it take to hit the ground?
(b) What is its speed when hitting the ground?

$$h = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 24}{9.81}} \approx 2.21 \text{ s}$$

$$v = gt = 9.81 \times 2.21 \quad \Rightarrow \quad v \approx 21.7 \text{ m s}^{-1}$$

to find final velocity, one can also use $v^2 = u^2 + 2as$:

$$v^2 = \cancelto{0}{u^2} + 2gh = 2 \times 9.81 \times 24 \quad \Rightarrow \quad v \approx 21.7 \text{ m s}^{-1}$$

Example 2.16 A photograph is taken for a small particle falling from rest. The photograph was taken at 0.400 s after the object is released. Since

[10] The reason for this constant acceleration of free fall will be elaborated in §3.5.

[11] Note that Maths will make you use 9.8 m s^{-2}

the particle is still moving when the photograph is being taken, the image is blurred. It is found that the blurred part corresponds to a length of 20.8 cm moved out by the particle, what is the time of exposure for the photograph?

from $t = 0$ to right before photo is taken:
$$s_1 = \frac{1}{2} g t_1^2 = \frac{1}{2} \times 9.81 \times 0.400^2 \approx 0.785 \text{ m}$$
from $t = 0$ to right after photo has been taken:
$$s_2 = s_1 + \Delta s = \frac{1}{2} g t_2^2 \quad \Rightarrow \quad t_2 = \sqrt{\frac{2(s_1+\Delta s)}{g}} = \sqrt{\frac{2\times(0.785+0.208)}{9.81}} \approx 0.450 \text{ s}$$
time of exposure: $\Delta t = t_2 - t_1 = 0.450 - 0.400 \approx 0.050$ s

Figure 2.1: Milk production at a dairy farm was at an all-time low, so the farmer decided to call on the local university for help. A team of the institutions' top scientists was assembled to work on the problem. The team of scientists visited the farm and took extensive data carefully examining every step of the production process. The leader of the team, a theoretical physicist, offered to write the report. A week later the physicist returned to the farm, saying to the farmer, "I have found a solution for your problem, but it only works for spherical cows in vacuum."

Upward projection

Like a freely-falling object, an object thrown upwards experiences the same constant downward acceleration $a = g \approx 9.81$ m s^{-2}, as long as drag forces can be ignored. Remember that we pretend that we live in a "Perfect Physics World"™for most of the A-level course. No air resistance, no friction, no inconvenient complexities.
Note that initial velocity u is upwards, but acceleration a is downwards so we will have different signs for u and a in the equations, conventionally, positive direction is taken as same direction as initial velocity. In our case, positive direction is upwards, the acceleration is then negative: $a = -g$ so the velocity-time relation and displacement-time relation are:

$$v = u - gt \qquad s = ut - \frac{1}{2}gt^2$$

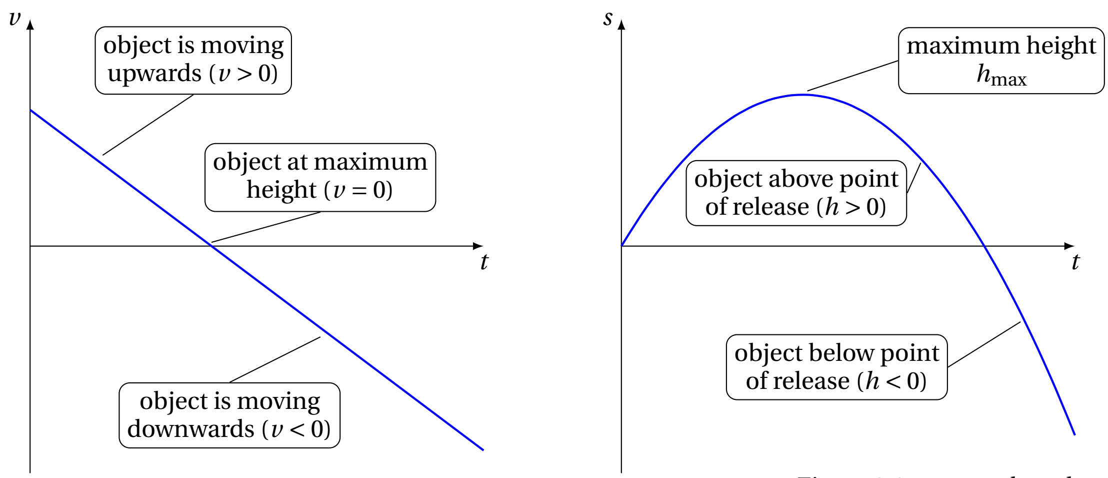


Figure 2.6: v-t graph and s-t graph for upward projectile motion

The sign of v now tells direction of motion, $v > 0$ means object is moving upwards, $v < 0$ means it has reversed direction and starts falling, in particular, an object attains greatest height when $v = 0$.
The sign of s gives whether object is at a higher or lower position with respect to point of release.

– $s > 0$ means the object is above the position from which it is projected
– $s < 0$ means it is now below the point of release

Example 2.17 A ball is projected vertically upwards at 12 m s^{-1}. Air resistance is negligible. (a) Find the time taken for the ball to reach the highest position. (b) Find the greatest height.

> maximum height is reached when $v = 0$, so
> $$v = u - gt = 0 \quad \Rightarrow \quad t = \frac{u}{g} = \frac{12}{9.81} \approx 1.22 \text{ s}$$
> $$H_{\max} = ut - \frac{1}{2}gt^2 = 12 \times 1.22 - \frac{1}{2} \times 9.81 \times 1.22^2 \approx 7.34 \text{ m}$$
> It is also possible to use $v^2 - u^2 = 2as$ to find $H_{\max}$:
> $$0^2 - u^2 = 2(-g)H_{\max} \quad \Rightarrow \quad H_{\max} = \frac{u^2}{2g} = \frac{12^2}{2 \times 9.81} \approx 7.34 \text{ m}$$

Example 2.18 A stone is thrown vertically upwards with an initial velocity of 14.0 m s^{-1} from the edge of a cliff that is 35 m from the sea below. (a) Find the speed at which it hits the sea.
(b) Find the time taken for the stone to hit the sea. [12]

[12] Note that we have substituted $a = -g$ since acceleration of free fall always points downwards, and $s = -35$ m since sea is below point of release. Also final velocity when hitting water is downwards, which should take a negative sign, so we discarded the positive solution for v.

> Take positive direction to point upwards, we use $v^2 - u^2 = 2as$ to find $v^2 = 14.0^2 + 2 \times (-9.81) \times (-35) \approx 883 \text{ m}^2 \text{ s}^{-2} \Rightarrow v \approx -29.7 \text{ m s}^{-1}$. To find time, we can use $v = u - gt$, hence: $t = \frac{v-u}{-g} = \frac{-29.7-14.0}{-9.81} \approx 4.46$ s.
> One can also attempt $s = ut - \frac{1}{2}gt^2$, this leads to the equation: $-35 = 14.0t - \frac{1}{2} \times 9.81t^2$.
> This quadratic equation in t gives two roots: $t_1 \approx 4.46$ s, and $t_2 \approx -1.60$ s.
> Note that a negative root should be discarded since it means stones hits the sea before it is thrown so time taken for stone to hit the sea is $t \approx 4.46$ s

Motion in two dimensions – projectile motion

For projectile motion, we assume no air resistance and no other forces gravity causes a constant acceleration of free fall that acts vertically downwards. The curved path of a projectile is the combination of its *horizontal* and *vertical* motion.

– horizontally: no acceleration, so horizontal component of velocity v_x = constant

– vertically: constant acceleration, vertical component of velocity v_y varies over time

As a consequence, a projectile would follows a *parabolic* path as it travels[13]

[13] You may be able to prove this statement in Question 2.24.You may try to prove this statement with a bit algebra.

Let's consider a projectile launched at initial velocity u at angle θ to the horizontal:

Horizontally, projectile maintains a constant velocity, so:

$$v_x = u_x \qquad x = u_x t$$

Where $u_x = u\cos\theta$ is horizontal component of initial velocity vertically, if

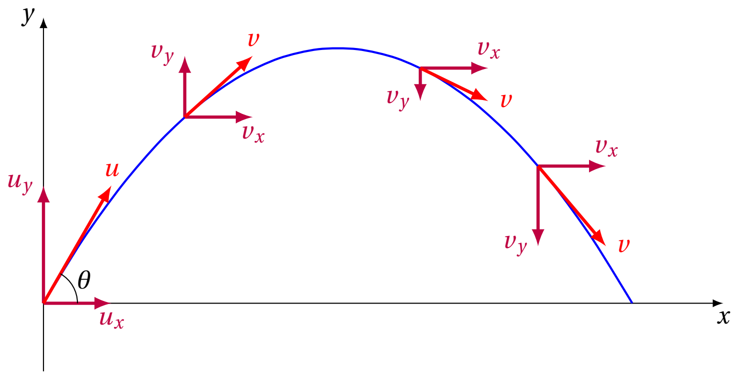


Figure 2.7: components of the velocity of a projectile at different points along its path

upward direction is taken to be positive, then acceleration $a = -g$, so

$$v_y = u_y + at = u_y - gt \qquad y = u_y t + \frac{1}{2}at^2 = u_y t - \frac{1}{2}gt^2$$

where $u_y = u\sin\theta$ is vertical component of initial velocity - components can be combined to give resultant velocity or resultant displacement:

$$v = \sqrt{v_x^2 + v_y^2} \qquad s = \sqrt{x^2 + y^2}$$

Maximum height reached by an projectile

When a projectile reaches the highest position, its instantaneous vertical velocity becomes zero, we can then find the time it takes to attain this maximum height.

$$v_y = u\sin\theta - gt = 0 \quad \Rightarrow \quad t = \frac{u\sin\theta}{g}$$

To find $H_{\max}$, one can use either equation (2.2) or (2.3)

$$H_{\max} = \frac{1}{2}(u_y + v_y)t = \frac{1}{2}u_y t = \frac{1}{2} \times u\sin\theta \times \frac{u\sin\theta}{g} = \frac{u^2\sin^2\theta}{2g}$$

$$H_{\max} = u_y t - \frac{1}{2}gt^2 = u\sin\theta \times \frac{u\sin\theta}{g} - \frac{g}{2} \times \left(\frac{u\sin\theta}{g}\right)^2 = \frac{u^2\sin^2\theta}{2g}$$

For any given initial speed u, the greater the angle of projection, the higher the object will reach. In the extremal case where $\theta = 90°$, it simply becomes an upwardly projected motion.

Airborne time and horizontal range of an projectile

A ball projected from the ground will first rise in height but it will eventually fall to the ground due to the gravitational pull after a period of time T. When it lands, its vertical displacement is zero, so:

$$Y = u_y T - \frac{1}{2}gT^2 = u\sin\theta T - \frac{1}{2}gT^2 = 0 \quad \Rightarrow \quad T = \frac{2u\sin\theta}{g}$$

The horizontal range is given by:

$$X = u_x T = u\cos\theta \times \frac{2u\sin\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g} \quad \Rightarrow \quad X = \frac{u^2\sin 2\theta}{g}$$

Where in the last step the trigonometric identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$ has been used.

For a given initial speed u, a projectile launched at a greater angle will stay in the air for longer time, the greatest airborne time is obtained if object is projected straight up, i.e., $\theta = 90°$. For a given initial speed, horizontal range of projectile depends on angle θ of projection, so to obtain the greatest horizontal range, two things are required:

– sufficiently large horizontal velocity v_x

– sufficiently long time T staying in the air

however, a larger v_x requires a smaller θ, which means a shorter airborne time T, therefore, there is a compromise between the two. The optimal angle should be neither be too large nor too small, and can be shown to be 45° [14]

[14] You could prove this algebraicly if you were so inclined

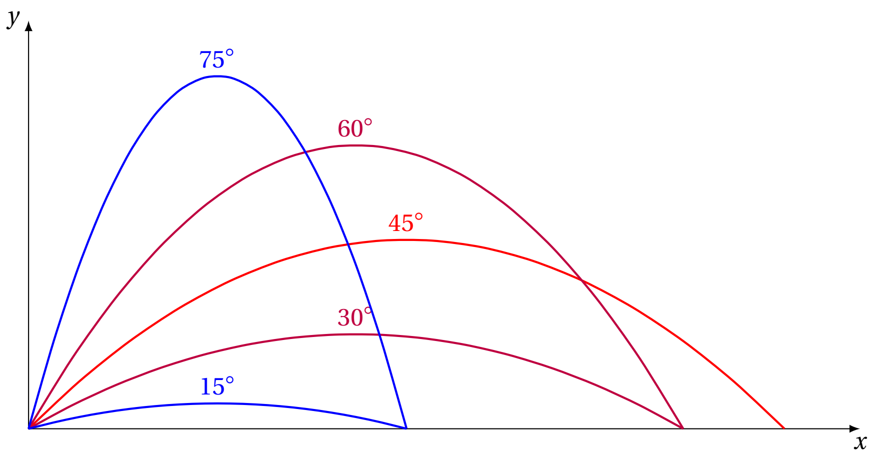


Figure 2.8: trajectories of projectiles launched at the same speed but different angles

Example 2.19 A ball is thrown from a point O at 15 m s^{-1} at an angle of 40° to the horizontal. The ball reaches its highest position at point P. Ignore the effects of air resistance. (a) How long does it take to reach P? (b) What is the magnitude of the displacement OP?

At the highest point: $v_y = u_y - gt = 0 \quad \Rightarrow \quad t = \frac{u\sin\theta}{g} = \frac{15\times\sin 40°}{9.81} \approx 0.983$ s

Vertical displacement: $y = u_y t - \frac{1}{2}gt^2 = 15\sin 40° \times 0.983 - \frac{1}{2}\times 9.81 \times 0.983^2 \approx 4.74$ m

Horizontal displacement: $x = u_x t = 15\cos 40° \times 0.983 \approx 11.3$ m

Resultant displacement: $|OP| = \sqrt{x^2+y^2} = \sqrt{11.3^2+4.74^2} \approx 12.2$ m

Example 2.20 A small object is horizontally projected at 7.20 ms^{-1} from a surface at a height of $h = 1.2$ m above the ground. Assume there is no air resistance. (a) What is the time taken for the object to hit the ground? (b) What is the horizontal range? (c) Find the velocity at which the object hits the ground.

Vertically, take downward as positive: $h = \cancelto{0}{u_y t} + \frac{1}{2} g t^2 \Rightarrow t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1.2}{9.81}} \approx 0.495$ s
Horizontal range: $x = u_x t = 7.20 \times 0.495 \approx 3.56$ m
Final vertical velocity: $v_y = \cancelto{0}{u_y} + g t = 9.81 \times 0.495 \approx 4.85\ \mathrm{m\,s^{-1}}$
Magnitude of resultant velocity: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{7.20^2 + 4.85^2} \approx 8.68\ \mathrm{m\,s^{-1}}$
Angle to which resultant velocity makes with horizontal:
$\phi = \tan^{-1} \frac{v_y}{v_x} = \tan^{-1} \frac{4.85}{7.20} \approx 34°$

End-of-chapter questions

Kinematic quantities

Question 2.1 What is the distance covered for a car that travels half a lap along a circular path of radius of 200 m. What about the displacement?
Question 2.2 A ball is released from a height of 2.0 m above the ground. It bounces vertically for quite a number of times before coming to rest. (a) State the change of displacement for the ball. (b) Explain how the distance travelled is different from the change in displacement.
Question 2.3 For an athlete running around a track for many laps, suggest how his average velocity could be zero?
Question 2.4 A car travels 2400 m east in 3.0 minutes, then takes a left turn, and then travels 700 m north in 1.5 minutes. What is the average speed and the average velocity for this journey?
Question 2.5 Is it possible for an object moving at a steady speed to have acceleration?

Motion graphs

Question 2.6 For the v-t graph given in Example 2.11, sketch the a-t graph for this motion.
Question 2.7 If the tangent of a displacement-time graph at one particular instant is sloping downwards, what does that imply about the velocity at that instant?
Question 2.8 A vehicle initially travels at a steady speed of 15 $\mathrm{m\,s^{-1}}$. It accelerates uniformly for 10 s to reach a higher speed of 20 $\mathrm{m\,s^{-1}}$. It maintains at this speed for 20 s, and then decelerates uniformly to a stop in the last 10 s. (a) Sketch the velocity-time graph for this motion. (b) Sketch the acceleration-time graph. (c) Find the distance travelled during the 40 s.

Linear motion with constant velocity

Question 2.9 Sonar is a technique that uses sound waves to detect objects. It can be used to measure the depth of the seabed. Given that speed of sound in water is 1500 m s^{-1}, and reflected waves sent from a submarine are detected 0.50s after they are transmitted. How deep is the water below the submarine?
Question 2.10 Given that the speed of sound in air is 340 m s^{-1} and the speed of light in air is 3.0×10^8 m s^{-1}. If a person hears the sound of a thunder 5.0 seconds after seeing a lightning flash, how far away from this person is did the lightning strike?

Linear motion with constant acceleration

Question 2.11 A train initially travels at a speed of 40 m s^{-1}. It starts to decelerate at 0.50 m s^{-2}. (a) What is the distance travelled in 50 s? (b) When it comes to a stop, how far out has it travelled?
Question 2.12 A vehicle moving at 14 m s^{-1} accelerate uniformly to 26 m s^{-1} in 6.0 s. (a) What is the average velocity during this time? (b) What is the acceleration during this time (c) What is the distance travelled by the vehicle? (d) The vehicle then braked with constant deceleration to stop in another 8.0 s. What is the distance travelled during the time when brakes are applied?

Free fall

Question 2.13 Two balls are dropped from rest from the same height. The second ball is released 0.80 s after the first one. What is their separation 1.5 s after the second ball is dropped?
Question 2.14 A golf ball is dropped from the top of a tower of height 30 m. The ball falls from rest and air resistance is negligible. What time is taken for the ball to fall (a) the first 10 m from rest, (b) the last 10 m to the ground?
Question 2.15 The acceleration of free fall on Pluto is about one-fifteenth of that on Earth. If it takes a time of T for a rock to fall from rest a distance of S, what is the time taken, in terms of T, for a rock to fall from rest through the same distance S on Pluto?
Question 2.16 In an experiment is carried out to determine the acceleration of free fall g using a falling body. The body is released from rest from a height of h, the time taken t for it to hit the floor is measured. (a) Find the expression that can be used to calculate the value of g? (b) Suggest what could lead to an overestimation for the value of g.

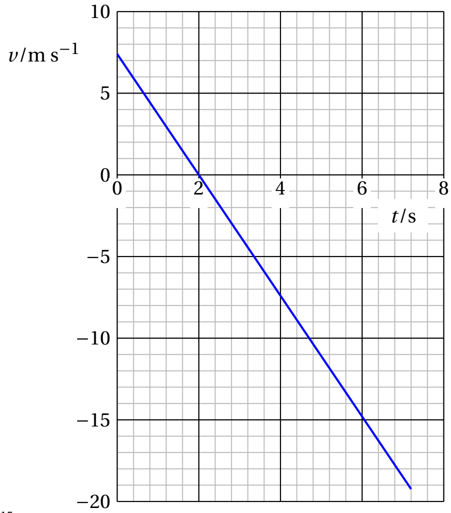
Upward projection

Question 2.17 A ball is tossed upwards with a speed of 9.0 m s^{-1}. (a) How long does it take to return to the same point if air resistance is negligible? (b) How does the return velocity compare with its initial velocity?
Question 2.18 Someone wants to toss a ball onto a platform that is at a height of 20 m above him. What is the minimum initial velocity needed to launch the ball?

Question 2.19 Someone standing at the top of a high building throws a ball straight up and another ball straight down with the same initial speed. Assume that air drag is negligible, which ball will have a greater speed when it hits the ground?

Question 2.20 In basketball games, hang time refers to the length of time a player stays in the air after jumping from the floor. (a) It is reported that Michael Jordan is able to stay in the air for $T = 1.0$ s to do his slam-dunk tricks, estimate how high he can jump. (b) The acceleration of fee fall on the Moon is about one-sixth of that on Earth. What is the hang time of Michael Jordan if he takes off from the surface of the Moon?

Question 2.21 Mark Watney[15] stands at the edge of a cliff on the Mars and throws a rock vertically upwards with a speed of 7.4 m s^{-1}. The graph shows the variation with the time t of the rock's velocity v. (a) What is the acceleration of free fall on the Mars? (b) When does the rock reach the maximum height? (c) What is the height above the base of the cliff the moment when the rock is thrown? (d) What is the maximum height above the base of the cliff to which the rock rises? (e) What is the total distance travelled by the rock before it strikes the ground?



[15] A fictional character in the science fiction movie *The Martian* (2015) based on the novel of the same name written by *Andy Weir*.

Projectile motion

Question 2.22 A ball rolls off a table and lands at a position of a horizontal distance of 1.2 m from the table. The table is 0.95 m high. Find the speed at which the ball leaves the table.

Question 2.23 A ball is kicked from the ground towards a vertical barrier. The barrier is at a horizontal distance of 18 m from the initial position of the ball. The ball strikes the barrier after 1.5 s at a height of 2.5 m above the ground. (a) Find the magnitude and the direction of the initial velocity. (b) Find the magnitude and the direction of the velocity at which the ball hits the barrier.

Question 2.24 When a particle is launched from the origin at an angle θ with the horizontal at a speed of u, show that its trajectory is a parabola given by the equation: $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$.

Question 2.25 Two golf players each hit a ball at the same speed. One at 30° with the horizontal, the other at 60°. Which ball hits the ground first? Which ball goes farther?

Question 2.26 (a) State the difference between the displacement of a projectile and the distance it travels. (b) Suggest in what situation a projectile's displacement could have the same magnitude as the distance.

Question 2.27 An archer always aims slight higher than the distant target that she wants to hit. Why isn't the bow lined up such that it points exactly at the target?

3 Force & Motion

Force & motion: an introduction

In physics, a force appears when two bodies interact with one another - you will encounter various types of forces in this course, some of which are:

- *weight*: the gravitational attraction acting on any object exerted by the earth
- *tension*: a force in a string, a rope, a chain, etc. when it is being pulled
- *normal contact*: when a body's surface is compressed, there reacts a normal contact force[1]
- *friction*: a force that resists relative motion when two surfaces tend to slide over one another
- *resistance*: also called drag force, this is experienced when a body travels through a medium
- *upthrust*: an upward force acting on an object immersed in a fluid
- *electric force*: an attractive or repulsive interaction between electrically charged objects
- *magnetic force*: an attractive or repulsive interaction between magnets or electric currents

Detailed features of these forces follow later in the notes.

A force can produce various effects to the object, the effect could be:

- an increase/decrease in speed
- a change in the direction of motion
- causing the object to rotate
- a change in shape of the object

In the next few chapters, we will be looking into each of these aspects.

When more than one force act on a body, it is useful to find their *resultant*, or the *net force*

> **resultant force**, or **net force**, is a single force that has the same effect as all forces acting on an object combined. The emphvector sum of all of the individual forces gives the resultant force.

In this chapter, we will study the dynamics of *point masses* - a **point mass** is an idealization that the object has a mass but does not take up any space, furthermore an object treated as a point mass has a position specified with a geometric point in space. This is, of course, another simplification when size, shape, rotation, or structure of object are not important (Spherical cows..).

[1] Examples of normal contact force are support force that stops a desk from sinking into the ground, the impact on a football when you kick it, etc.

Figure 3.1: Perfect physics world strikes again...

Newton's laws of motion

Newton's laws of motion[2] are three laws that form the basis of classical mechanics, they describe the relationships between motion of an object and forces acting on it.

[2] These three laws were first addressed by *Isaac Newton* in his famous work *Mathematical Principles of Natural Philosophy*, or simply the *Principia*. The three-volume work was first published in 1687, and was soon recognised as one of the most important works in the history of science. Apart the from the three laws that laid the foundations for classical mechanics, the *Principia* also stated *the law of gravitation*, and accounted for planetary orbits and tides and other phenomena.

Newton's first law

> **Newton's first law** states that an object continues in its state of rest or uniform motion at constant velocity if there is no resultant force acting

Any object 'dislikes' any change to its state of motion, uniform motion tends to persist forever, this tendency to resist changes in motion is called the **inertia**, in some textbooks, Newton's first law is also called *the law of inertia*. If there is no change in state of motion, the object is said to be in **equilibrium**.

Newton's Second law

If resultant force is non-zero, velocity of the object will change, i.e., force produces acceleration.

> **Newton's second law** states that the acceleration is proportional to the resultant force and inversely proportional to the mass of the object. Symbolically, we write $a \propto \frac{F_{\text{net}}}{m}$

With consistent units of measure, this proportionality can be written as an exact equation:

$$a = \frac{F_{\text{net}}}{m} \quad \text{or}$$

$$\boxed{F_{\text{net}} = ma} \tag{3.1}$$

– The SI unit of measurement for force F is **newton** (N)
A force of one newton acting a body of 1 kg produces an acceleration of 1 m s^{-2}, acceleration produced is always in same direction of the net force. For the same force, an object of greater mass has a smaller acceleration, hence mass is a measure of the *inertia* of this object in response to a net force.
The way that the mass of an object is defined from the point of view of Newton's laws can be stated as[3]:

> **Mass** is an intrinsic property of a body to resist any change in its state of motion.

Example 3.1 A box of 6.0 kg is being pushed along a horizontal surface with a force of 30 N. The resistive force acting is 21 N. What is the acceleration of the box?

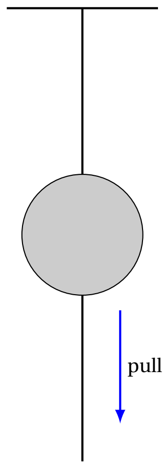
[3] The concept of mass can be defined in many different ways. You might be familiar with the definition for mass as the amount of matter an object possesses. I personally think this definition is a bit vague and does not tell you anything new. Thinking of mass as a measure of inertia surely brings more insights. Mass also tells the strength at which an object interacts with other bodies through the gravitational attraction. As you will see later, from the view of Albert Einstein, it is also possible to think of mass as a form of energy.

$$F_{net} = F - f = ma \quad \Rightarrow \quad a = \frac{F - f}{m} = \frac{30 - 21}{6.0} = 1.5\ \mathrm{m\,s^{-2}}$$

Example 3.2 A car of mass 800 kg is travelling at a speed of 20 m $\mathrm{s^{-1}}$. The driver then operates the brake pedal so a braking force of 2000 N gradually brings the car to stop. (a) What is the deceleration for the car? (b) What is the stopping distance?

Using Newton's second law and noticing braking force acts opposite to direction of motion:

$$F_{net} = ma \quad \Rightarrow \quad -2000 = 800 \times a \quad \Rightarrow \quad a = -2.5\ \mathrm{m\,s^{-2}}$$
$$2as = v^2 - u^2 \quad \Rightarrow \quad s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 20^2}{2 \times (-2.5)} = 80\ \mathrm{m}$$



Example 3.3 A massive ball is suspended on a string. A second string is attached to the bottom of the ball. If one pulls the bottom string with a gradually increasing force, does the top string or the bottom string break first? What if the bottom string is jerked, which string breaks?

When tension gradually increases, system is always in equilibrium, the tensions in strings must have $T_{top} = T_{bottom} + mg$. The top string suffers a greater force, so it breaks first, however, when bottom string is jerked, the ball tends to remain at rest due to its large mass, preventing sudden change to the tension in top string so in this case bottom string is more likely to snap.

Example 3.4 Leviation of a slinky - When the top end of the slinky is dropped, the information of the tension change must propagate to the bottom end before both sides begin to fall; the top of an extended slinky will drop while the bottom initially remains in its original position, compressing the spring.

Newton's third law

Every force is part of a pair of interactions between one body and another, when one body exerts a force on another, the second body also exerts a reaction on the first.

Newton's third law, also called the **action-reaction principle**, states that action and reaction are always equal in magnitude, opposite in directions and of the same type.

Example 3.5 Suggest the action and reaction force in the following cases: (a) A person stands on a bathroom scale. (b) A helicopter hovers in air. (c) The earth orbits around the sun.

(a) The person exerts downward force on scale, scale exerts an upward reaction on the person.
(b) The rotors of helicopter push air downwards, air exerts an upward force on helicopter.
(c) The Sun pulls the earth through gravitational attraction, the Earth also attracts the sun in return.

Force analysis & free-body diagrams

When doing mechanics problems, it is necessary to find all forces applied upon an object, to visualise all these forces, it is helpful to draw a **free-body diagram** (FBD). A FBD shows a simplified version of the body with arrows indicating forces applied, it is recommended to follow the routine stated below when solving a mechanics problem:

(1) draw a FBD for the object in the problem
(2) resolve and find the resultant force with aid of the FBD
(3) apply Newton's laws to write down the equation of motion for the object
(4) solve the equation(s) to find acceleration
(5) use kinematic relations to deduce information about motion of the object

Types of forces

Weight

All objects exert attractive forces of gravity upon each other.[4] The **weight** of a body is due to the gravitational pull from our planet – the Earth.
The weight W of any object is proportional to its mass m:

$$W = mg$$

Where g is strength of the gravitational field, or the gravitational acceleration constant, The value of $g \approx 9.81\ \mathrm{N\,kg^{-1}}$ but this value for g does not hold in a satellite orbit, on Mars, near a black hole, etc.
The concept of weight is different from mass in many aspects:

- weight is a force, so it is a vector (always acting downwards still makes a direction)
  mass is a scalar, it has magnitude only
- weight is measured in newtons, mass is measured in kilograms
- weight of object depends on its mass but also strength of gravitational field
  mass is an intrinsic property of object, so does not depend on force fields
  same object can have different weights on different planets, but its mass will be the same[5]

[4] You will learn more about gravitational forces in Year 13.

[5] Here we do not take into account the effects of *relativity*. A clever student who has learned Einstein's theories might suggest the mass of the same object increases with its velocity.

Example 3.6 An astronaut finds that he weighs 300 N on the surface of Mars, where the gravitational field strength is known to be 3.7 N kg^{-1}. Find his mass and hence estimate his weight if he returns to his home on the Earth.

Mass of astronaut: $m = \frac{W_M}{g_M} = \frac{300}{3.7} \approx 81.1$ kg
Weight on earth: $W_E = mg_E = 81.1 \times 9.81 \approx 795$ N

Free fall

All things on the earth fall because of the force of gravity, if we ignore the restraints such as air resistance and upthrust force on a falling object, say the object is under the influence of gravity only, then the object is in a state called **free fall**. Assuming the object is subject to gravity only, the resultant force is simply its weight, applying the Newton's second law, we have $F_{\text{net}} = W \quad\Rightarrow\quad ma = mg$ so acceleration of the freely-falling object is[6]:

$$a = g$$

The above shows us that **acceleration due to free fall** is simply equal to field strength g, so any object, regardless of its mass, has same acceleration due to free fall.

[6] In the derivation, the mass terms cancel out. Rigorously speaking, these are two different masses. One is the measure of inertia, and the other is a measure of gravitational force. It is experimentally found that the inertia mass and the gravitational mass are equal. The fact that the two masses are equal has profound reasons. We have shown here acceleration of free fall equals gravitational field strength, but Albert Einstein's suggests that it is actually impossible to distinguish between a uniform acceleration and a uniform gravitational field. This idealies at the heart of his *general theory of relativity*. Those who are interested in this topic are recommended to start from here and do some online researches.
In §2.14 and §2.18, the statement that acceleration of free fall is constant in absence of air resistance was asserted without further explanation. Now you know why.

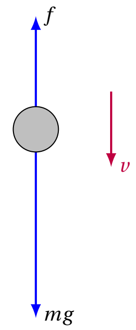
Drag

When a body moves through air, water or any fluid, it experiences resistance called drag force, factors that determine the value of fluid drag include:

- relative speed of the object to the fluid ($v \nearrow \Rightarrow f \nearrow$)
- cross section of the object ($A \nearrow \Rightarrow f \nearrow$)
- shape of the object (streamlined shape has smaller drag)
- density of the fluid ($\rho \nearrow \Rightarrow f \nearrow$)

but what determines the drag force is a complicated issue[7] in short, the drag force always acts in a direction to oppose relative motion of object through fluid.

[7] There are a few empirical formula for drag force, each of which is accurate under certain conditions.
For an object moving through a fluid at low speeds (*laminar flow*, no turbulence occurs), the resistance it experiences is proportional to its speed: $f = bv$, where b is some constant which depends on fluid viscosity and the effective cross-sectional area of the object.
If objects are moving at relative high speeds through the fluid such that *turbulence* is produced behind the object, drag force is proportional to the speed squared: $f = \frac{1}{2}\rho C_D A v^2$, where ρ is the fluid's density, A is the cross-sectional area, C_D is a dimensionless quantity called the drag coefficient.



Free fall through air

Let's consider an object falling through air from a very high tower, the forces acting are weight and air resistance (shown in the free-body diagram). The equation of motion for this falling object is:

$$F_{\text{net}} = mg - f = ma$$

As v increases, air resistance f increases, so net force F_{net} decreases, this means acceleration a would decrease as object falls - i.e., speed will increase at a decreasing rate during the fall [8]
at low speeds, air resistance is negligible, so $F_{\text{net}} = ma \approx mg$, acceleration of object at start of the fall is similar to g but as v increases, acceleration decreases so a becomes less than g. After a sufficiently long time, acceleration gradually decreases to zero, velocity gradually increases and

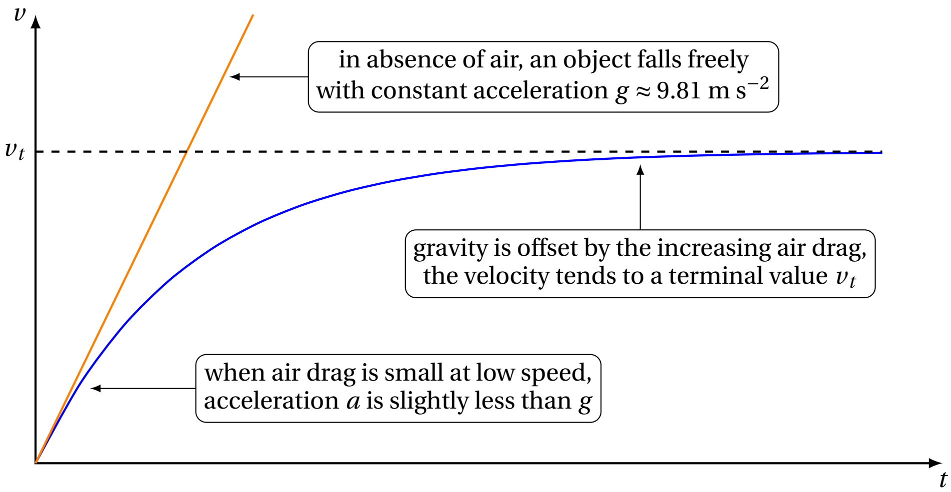


Figure 3.1: variation of velocity for a falling object through air

tends to a maximum value. At this stage, equilibrium is restored: $f = mg$, object no longer accelerates, this constant final velocity is known as the **terminal Velocity**

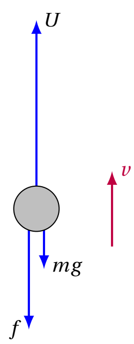
Example 3.7 An object of 5.0 kg falls through the atmosphere from a very high altitude. After some time, it falls at a constant speed of 70 m s^{-1}. Assume there is no significant change in gravitational field during the fall and the air resistance is proportional to speed: $f = bv$. (a) Find the value of the coefficient k. (b) Find the acceleration of the object when it is falling at 30 m s^{-1}.

equilibrium between weight and air drag when falling at terminal speed, so

$$mg = bv_t \quad \Rightarrow \quad b = \frac{mg}{v_t} = \frac{5.0 \times 9.81}{70} \approx 0.70 \text{ kg s}^{-1}$$

at any instant, equation of motion is: $F_{\text{net}} = ma = mg - bv$
at 30 m s^{-1}, acceleration is: $a = \frac{mg-bv}{m} = \frac{5.0\times9.81-0.70\times30}{5.0} \approx$ 5.6 m s^{-2}

Bubble rising in a liquid



let's now consider bubbles formed at the bottom of a fizzy drink. The forces acting on bubble are weight, water resistance and upthrust. The equation of motion for the rising bubble is:

$$F_{\text{net}} = U - mg - f = ma$$

As bubble moves faster, f increases, then F_{net} decreases, so acceleration a would gradually decrease to zero as bubble rises. The speed of the bubble increases and reaches a maximum value at terminal speed, $a \to 0$, therefore one has:

$$U = f + mg$$

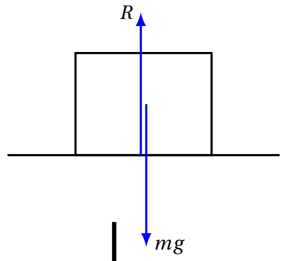
Normal contact

When two objects are in contact, the interaction between them is called the *contact force*. The **normal contact force** is the component of contact force perpendicular to the contacting surface.[9]
Of course, we know that there are actually only 4 forces in the universe so the origin of normal contact is the *electrostatic interaction* between atoms. It's convenient shorthand however to be able to talk about contact and non-contact forces and it's *close enough*.[10] When two objects are pressed against each other, surface atoms get close enough that the strong electrostatic repulsion between electron clouds of the atoms prevent them from penetrating through one another.

[9] By definition, "normal" is always at right angles to the surface in question. Cf. refraction and reflection.

[10] Pun intended.

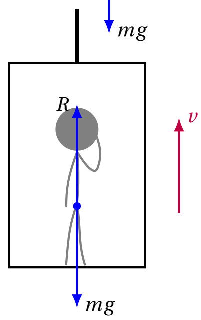
Example 3.8 A box of mass $m = 4.0$ kg is resting on a horizontal ground. What is the normal contact force acting?



Equilibrium between weight and normal contact, so

$$R - W = 0 \quad \Rightarrow \quad R = mg = 4.0 \times 9.81 \approx 39.2 \text{ N}$$

Example 3.9 A man of 80 kg stands in a lift. Find his apparent weight, i.e., the contact force, when the lift is (a) moving upwards at steady speed of 2.0 m s^{-1}, (b) accelerating upwards at 2.0 m s^{-2}, (c) moving upwards but slowing down at a deceleration of 1.5 m s^{-2}.



The forces acting on man are weight and normal contact, for either case, equation of motion for the man reads:

$$F_{\text{net}} = ma = R - mg$$

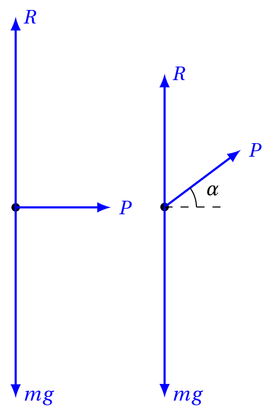
so normal contact force: $R = mg + ma$

when rising at steady speed, man is in equilibrium ($a = 0$), so: $R = mg = 80 \times 9.81 \approx 785$ N

when accelerating upwards ($a = +2.0$ m s^{-2}): $R = 80 \times 9.81 + 80 \times 2.0 \approx 945$ N

when decelerating upwards ($a = -1.5$ m s^{-2}): $R = 80 \times 9.81 + 80 \times (-1.5) \approx 665$ N

Example 3.10 A sleigh of mass 15 kg lies at rest on an icy ground. The surface is frictionless. A force P of 75 N is applied to the sleigh. Find the normal contact force and the acceleration of the sleigh if P is acting (a) horizontally, (b) at an angle α to the horizontal where $\tan \alpha = \frac{3}{4}$.



(a) (b)

Free-body diagrams for both cases are shown for both cases, no net force acts in vertical direction. The net force in horizontal direction provides acceleration

when P acts horizontally:

$R = mg = 15 \times 9.81 \approx 147$ N

$P = ma \quad \Rightarrow \quad a = \frac{P}{m} = \frac{75}{15} = 5.0$ m s^{-2}

when P acts at angle α: $R + P \sin \alpha = mg \quad \Rightarrow \quad R = 15 \times 9.81 - 75 \times \frac{4}{5} \approx 87$ N

$$P \cos \alpha = ma \quad \Rightarrow \quad a = \frac{P \cos \alpha}{m} = \frac{75 \times \frac{3}{5}}{15} = 3.0 \text{ m s}^{-2}$$

Friction

Friction is the component of contact force that is parallel to contact surfaces, when there is potential or actual sliding between surfaces, frictional force come into action

– for surfaces that *tend* to move relative to each other, **static friction** acts to oppose this tendency.

– if surfaces are already sliding over one another, then **kinetic, or dynamic friction** opposes this motion.[11]

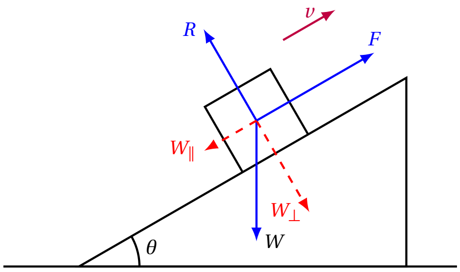
You could say that static friction f_S is self-adjusting, in that it arises as an $N3^{rd}$ force and will always be equal to the applied force - up until the moment the capacity for static friction is exceeded[12]. In other words, an object placed on a rough surface can stay at rest when acted by a small external force F it can do so because f_S equalises external force to maintain equilibrium, if no external force acts, then $f_S = 0$

Friction, on microscopic level, is an *electromagnetic force* in nature when two surfaces are in contact, irregularities on the surface touch each other surface atoms come very close and bonds are formed through electrostatic force in some sense, surface atoms get *cold welded* to each other when surfaces try to move relative to each other, this electrostatic weld is origin of friction.

[11] ⚠We dont need to know these in these terms - we often lump dynamic friction together with any other frictive forces, like air-resistance, and just call th resultant "Drag". Remember, a lot of Physics is knowing when to simplify.

[12] You'll learn about the coefficient of friction μ in maths

Inclined slopes



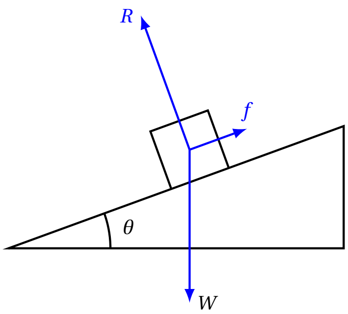
The inclined slope is probably the entry ticket into the business end of mechanics problems. There are *hundreds* of examples of this notorious problem to be found in any physics textbook and any exam paper on mechanics.

The problem is about a mass m placed on a plane inclined at angle θ to the horizontal, the mass could sit at rest on, slide down, or get pulled/-pushed up the plane. The motion of the mass could be affected by weight, friction, normal contact, or other forces.

They almost all have the same trick - resolve the forces in directions parallel and perpendicular to the slope because this allows you to use the equations of motion to work out motion up or down the slope. It is almost inevitable that you'll need to break weight into two components[13]

– component of weight parallel down the slope is: $W_{\parallel} = mg\sin\theta$
– component of weight perpendicular to slope is: $W_{\perp} = mg\cos\theta$

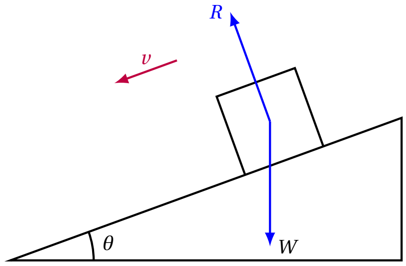
[13] We have already done that in Example 1.8.



Example 3.11 A block of mass m stays at rest on an inclined plane. The plane makes an angle θ with the horizontal. Find the normal contact force R and the frictional force f acting on the block.

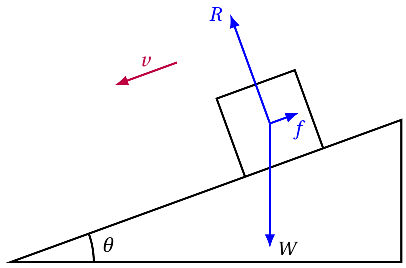
> The block is in equilibrium, so $F_{\text{net}} = 0$ in any direction.
> Parallel to slope: $f = W_{\parallel} \quad \Rightarrow \quad f = mg\sin\theta$
> Normal to slope: $R = W_{\perp} \quad \Rightarrow \quad R = mg\cos\theta$

Example 3.12 A block of mass m slides down a *smooth* slope. The angle of the slope to the horizontal is θ. Find the acceleration of the block.

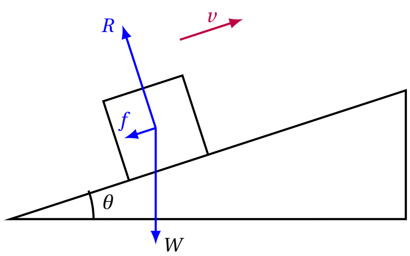


> only force acting along the slope is component of weight down the slope, so:
> $$F_{\text{net}} = ma = mg\sin\theta \quad \Rightarrow \quad a = g\sin\theta$$
> as $\theta \to 0$, $a \to 0$, this shows if plane becomes horizontal, the block simply stays put
> as $\theta \to 90°$, slope becomes vertical, block would undergo free fall, so naturally $a \to g$

Example 3.13 A block of mass 2.0 kg slides down a rough slope from rest. The slope is inclined at angle $\theta = 20°$ to the horizontal, and the block experiences a constant friction of 5.0 N. (a) What is the block's acceleration? (b) What is the distance travelled in 2.5 seconds?



> Resolving along slope:
> $$F_{\text{net}} = mg\sin\theta - f = ma$$
> $$a = \frac{2.0 \times 9.81 \times \sin 20° - 5.0}{2.0} \approx 0.855 \text{ m s}^{-2}$$
> Distance travelled: $s = \cancelto{0}{ut} + \frac{1}{2}at^2 = \frac{1}{2} \times 0.855 \times 2.5^2 \approx 2.67$ m



Example 3.14 A block of mass 3.0 kg is travelling up an inclined slope at an initial speed of 2.8 m s^{-1}. The slope makes an angle of 18° with the horizontal. A constant friction of 7.5 N acts on the block.
(a) What is the block's deceleration?
(b) How far does the block travel along the slope before its speed decreases to zero?
(c) Suggest whether the block could stay on the slope.

Resolving along slope (take direction of initial velocity as positive direction):

$$F_{\text{net}} = -mg\sin\theta - f = ma \quad \Rightarrow \quad a = \frac{-mg\sin\theta - f}{m} = \frac{-3.0\times 9.81\times\sin 18^\circ - 7.5}{3.0} \approx -5.53 \text{ m s}^{-2}$$

$$v^2 - u^2 = 2as \quad \Rightarrow \quad s = \frac{v^2-u^2}{2a} = \frac{0^2 - 2.8^2}{2\times(-5.5.3)} \approx 0.709 \text{ m}$$

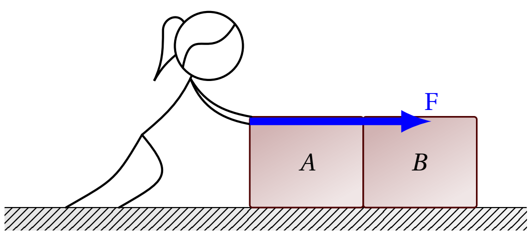
Note that component of weight down the slope is: $W_{\parallel} = mg\sin\theta = 3.0 \times 9.81 \times 18^\circ \approx 9.1$ N

$W_{\parallel} > f$, so friction is not enough to prevent block from sliding back down the slope

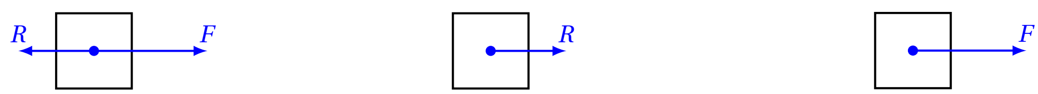
Many-body problems

The problems we have been dealing with so far only involve one body, however a lot of the "resolving vectors" type problems involve several objects that mutually interact. We can often exploit the fact that if you take any individual part of a system and look into the *internal* forces between the objects of interest you'll find that any force acting between objects *within* system, has an equal but opposite reaction force.
The system can also be treated as a whole and we can analyse the *net external force* acting on entire system and work out any resultant or combined force/acceleration/etc.

Example 3.15 Two boxes A and B are placed on a smooth surface. They are accelerated together by a horizontal force F as shown. Find the acceleration and the contact force between them.



free-body diagrams for A, B, and entire system are given below



A B A and B combined

[14]

[14] In fact, only two of the three equations are independent. You can easily check that adding the equation for A to that for B would produce the equation for the system. To solve the two unknowns for this problem, any two of the three equations shall do the job.

equations of motion can be written down for each free-body diagram and solved

$$\text{for } A\text{:} F - R = M_A a$$

$$\text{for } B\text{:} R = M_B a$$

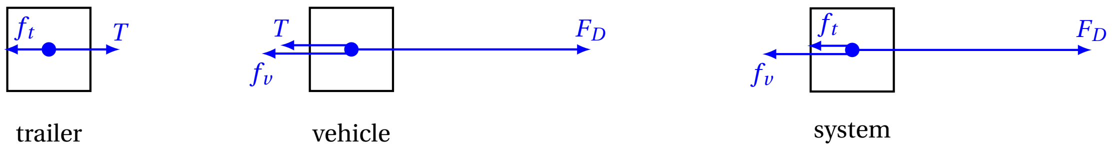
$$\text{for system:} F = (M_A + M_B) a$$

$$a = \frac{F}{M_A + M_B}$$

$$R = \frac{M_B}{M_A + M_B} F$$

Example 3.16 A vehicle of mass 1500 kg is towing a trailer of mass 500 kg by a light inextensible tow-bar. The engine of the vehicle exerts a driving force of 9600 N, and the tractor and the trailer experience resistances of 3600 N and 1800 N respectively. Find the acceleration of the vehicle and the tension in the tow-bar.

free-body diagrams for trailer, vehicle and entire system are given below



equations of motion can be written down for each free-body diagram:

$$\text{for trailer:} T - f_t = M_t a \quad \Rightarrow \quad T - 1800 = 500a$$

$$\text{for tractor:} F_D - f_v - T = M_v a \quad \Rightarrow \quad 9600 - 3600 - T = 1500a$$

$$\text{for system:} F_D - f_v - f_t = (M_v + M_t) a \quad \Rightarrow \quad = 2000a$$

solving simultaneous equations, we find

$$a = 2.1\ \text{m s}^{-2}, \quad \text{and} \quad T = 2850\ \text{N}$$

Tension in ropes

Tension, like the normal reaction force, is a useful shorthand that tells us a lot more about the situation than you might, as a new physicist, realise. The tension is a force along the length of a flexible medium, such as a rope or cable, it always acts both ways. To make that clearer, consider a person pulling a rope attached to a wall: The force on the rope from the person must be equal to the force on the person by the rope - from Newton's 3rd law. By the same logic, the force on the wall must be the same as the force exerted on the rope by the person.

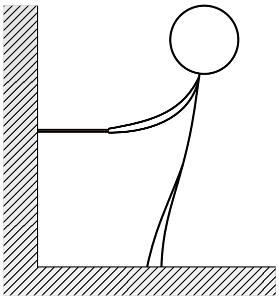


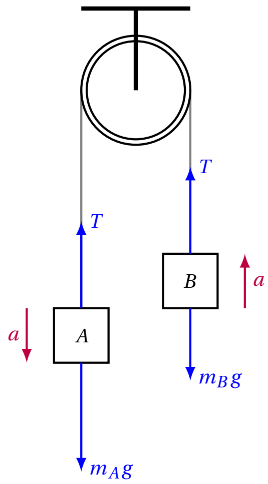
Figure 3.2: Pulling a rope from a wall

Pulleys

A *pulley* is basically a wheel that carries a string/rope/cable, in this section, we only consider pulleys whose axis of rotation is fixed, such pulleys can be used to change direction of tension in a taut string. We also assume pulleys to be *ideal*: they have no mass and no friction[15] for an ideal pulley, tensions on both sides are equal: $T_1 = T_2$.

[15] I really need a spherical cow icon, this will have to do:

Example 3.17 Two blocks of mass m_A and m_B ($m_A > m_B$) are joined together by a light inextensible string. The string passes over a smooth pulley as shown. The two blocks are suddenly released from rest. Find the acceleration of each block and the tension in the string.



Apply Newton's second law to each block:

$$\begin{cases} \text{for } A: & m_A g - T = m_A a \\ \text{for } B: & T - m_B g = m_B a \end{cases}$$

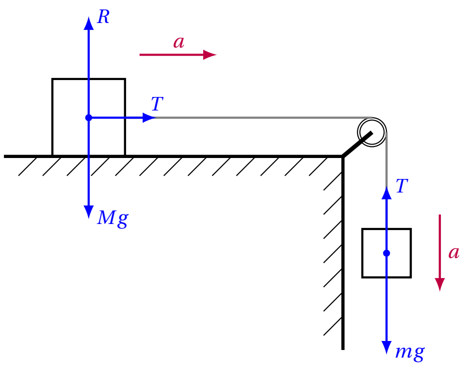
adding the two, one obtains equation of motion for whole system:

$$m_A g - m_B g = (m_A + m_B) a$$

solving these equations, we find

$$a = \frac{m_A - m_B}{m_A + m_B} g \qquad T = \frac{2 m_A m_B g}{m_A + m_B}$$

Example 3.18 A mass $M = 4.0$ kg is attached to a block of mass $m = 2.0$ kg through a light string which passes over a frictionless pulley as shown. When both masses are released, find the acceleration and the tension in the string.



Apply Newton's second law to each mass:

$$\begin{cases} \text{for } M: & T = Ma \\ \text{for } m: & mg - T = ma \end{cases}$$

adding the two equations, we have:

$$mg = (M + m) a$$

so acceleration is:

$$a = \frac{mg}{M+m} = \frac{2.0 \times 9.81}{4.0 + 2.0} = 3.27 \text{ m s}^{-2}$$

tension in string: $T = Ma = 4.0 \times 3.27 \approx 13.1$ N

End-of-chapter questions

Newton's first law

Question 3.1 A little girl tries to lift a luggage bag of mass 25 kg. She pulls upwards with a force of 150 N. The bag does not move. What is the normal reaction from the floor?
Question 3.2 To push a trolley around in a supermarket with constant velocity, you need to exert a steady force. How does this fact agree with Newton's first law, which suggests that motion with constant velocity requires no force?
Question 3.3 A worker is pulling a wagon of mass of 40 kg across a lawn at a constant velocity. He applies a force of 200 N at an angle of $15°$ above the horizontal. (a) Draw a free-body diagram for the wagon. (b) Find the frictional force. (c) Find the normal contact force.

Newton's second law

Question 3.4 (a) Forces of 3.0 N and 4.0 N act at right angles upon a mass of 160 g. What is the acceleration produced? (b) If the angle between the two forces are allowed to vary, what is the maximum possible acceleration they produce on the same mass? (c) What about the minimum possible acceleration?
Question 3.5 Explain why it becomes increasingly easier for an rocket to accelerate as it travels through space. (Hint: consider the fuel carried by the rocket.)
Question 3.6 Many cars are equipped with airbags which can inflate quickly in case of a collision event. Using Newton's second law, suggest why airbags could protect the driver and the passenger in the car during a car crash.
Question 3.7 A rocket of mass $30,000$ kg is launched vertically upwards at uniform acceleration of 1.6 m s^{-2}. What is the minimum thrust force required?
Question 3.8 A fire-fighter of mass 85 kg slides down a vertical pole. He descends through a distance of 6.0 m in 2.0 seconds. (a) Find the average acceleration. (b) Find the average frictional force acting on the fire-fighter.
Question 3.9 A trolley has mass m. A person needs to push the trolley with force F to produce an acceleration of a, and with force $2F$ to produce an acceleration of $3a$. Find, in terms of m and a, the constant resistive force opposing the trolley's motion.
Question 3.10 A girl stands onto a bathroom scale and finds the reading is 35.0 kg. She then takes the scale into a lift, what mass reading would she observe if the lift (a) is going down at a constant speed, (b) is accelerating downwards at 2.1 m s^{-2}?
Question 3.11 A pirate finds a box of gold coins at the bottom of a lake. The box and its contents have a total mass of 40 kg. The pirate pulls on the box by means of a cable, so that the box is made to rise vertically through the water. Meanwhile, the flow of water creates a constant horizontal force on the box, and the upthrust on the box is known to be 150

N. At one instant, the pirate applies a force of 380 N at an angle of 25° to the upward vertical, and the acceleration of the box is found to be 0.80 m s^{-2}. Assume all the forces acting are coplanar. (a) Draw a free-body diagram for the box. (b) Find the horizontal force due to water flow. (c) Find the drag force on the box.

Newton's third law

Question 3.12 A book placed on your desk experiences two forces: its weight and the support force. Identify the associated reaction forces.
Question 3.13 A student deduces that a rocket travelling in space can never accelerate because the propelling force acting on the rocket is cancelled by an equal and opposite force. Explain why this statement is incorrect.
Question 3.14 A U-shaped magnet lies on a top-pan balance and a mass reading of 180 g is registered. A current-carrying wire is then placed above the magnet. The wire experiences an additional force of 0.30 N that acts upwards. What is the mass reading on the balance?

Terminal velocity

Question 3.15 A light ball and a heavy ball of the same size are released from a very high tower, state and explain whether they will reach the ground at the same time.
Question 3.16 A stone is dropped from rest from a high tower. Air resistance is not negligible as the stone reaches terminal speed. Sketch two separate graphs to show the variation of its displacement and acceleration with time.
Question 3.17 How does the terminal speed of a parachutist before opening the parachute compare to that after? Explain your reasons.
Question 3.18 A ball is thrown horizontally from the top of a cliff. Effects of air resistance cannot be neglected. What happens to the horizontal and vertical components of the ball's velocity?
Question 3.19 A small sphere of mass 20.0 g is dropped from rest in a viscous liquid. When the sphere is moving at a speed of v, the viscous drag has a magnitude of $f = \alpha v^2$, where $\alpha = 14$ kg m^{-1}. (a) What is the sphere's acceleration at the instant when it is released? (b) What is the acceleration when it is moving at 5.0 cm s^{-1}? (c) What is the terminal velocity?
Question 3.20 A stone is thrown with some initial velocity at an angle to the horizontal. Sketch on the same graph the path of the stone if (a) air resistance is negligible, (b) air resistance is significant.

Inclined slopes

Question 3.21 A 3.0 kg mass is placed on an inclined plane and it does not move. Given that the normal contact force acting on it is 28.0 N. (a) Find the angle of the plane to the horizontal. (b) Find the frictional force acting on the mass.
Question 3.22 A small mass slides down a frictionless slope with an

acceleration of 2.8 m s^{-2}. Determine the angle that the slope makes with the horizontal.

Question 3.23 A car of mass 1400 kg is moving up a slope at a constant velocity of 13.5 m s^{-1}. The slope makes an angle of 6.0° to the horizontal. Total resistive force of 650 N acts on the car. What is the driving force required to push the car up the slope?

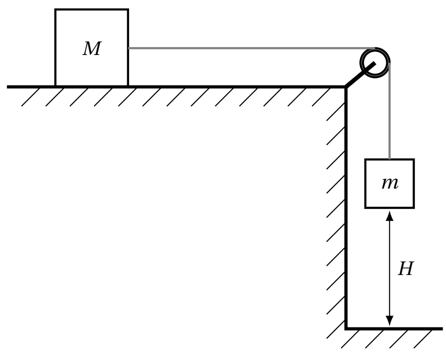
Question 3.24 A shopping trolley somehow loses control and runs down a straight slope from rest. The slope makes an angle of 3.0° to the horizontal. The resistive force acting on the trolley is a constant 15 N. The trolley and its contents have a total mass of 40 kg. (a) Find the acceleration of the trolley. (b) Determine the time for the trolley to travel a distance of 4.0 m along the the slope. (c) Suggest why the slope in shopping malls are not made any steeper.

Question 3.25 A heavy log of mass 240 kg is initially placed at a point P at the bottom of a slope. A motor drags the log up the slope through a cable. The slope is inclined at an angle of 16° to the horizontal. The motor provides a tension of 1200 N parallel to the slope. The friction that acts on the log is a constant 450 N. (a) Find the acceleration of the log. (b) Find the time taken to pull the log through a distance of 8.0 m to a point Q. (c) Find the velocity of the log at Q. (d) The cable breaks when the log reaches Q, find the distance moved beyond Q until the log's speed becomes zero. (e) The log will then slide back down the slope. Find the time for the log to return to its starting position. (f) Sketch a v-t graph for the log from the start at P until it returns to P.

Many-body problems

Question 3.26 Block A of mass 5.0 kg is connected by means of a light string to block B of mass 3.0 kg. The two blocks are placed on a horizontal table. A force of 30 N is applied to pull on block A. Given that the friction on each block is 30% of its own weight. (a) Find the acceleration of the blocks. (b) Find the tension in the string.

Question 3.27 A box of mass $M = 3.6$ kg rests on a horizontal, rough surface. The box is connected to a block of mass $m = 2.0$ kg through a light cord that passes over a frictionless pulley as shown. The box is released from rest. Given that the box experiences a frictional force of 12 N and the block is initially at a height of $H = 0.80m$ above the floor. (a) Find the acceleration of the block. (b) Determine the time taken for the block to hit the floor.



4 Mechanical Equilibrium

In this chapter, we will study the mechanical equilibrium of objects. We often consider mechanical problems in the context of *point objects*, in the case of a dimensionless point, there cannot be any turning moment and zero resultant force suffices for equilibrium. For *rigid bodies*, any force many produce a turning effect called a *moment*.
It follows then, that rigid bodies must satisfy another condition to stay in equilibrium...

Moment of force

The **moment** of a force, is defined as the product of the force and the perpendicular distance from the pivot to the line of action. The word **torque** is often used instead of moment, especially when we're ~~torquing~~ talking about the dynamic rotation of a motor and a change in angular momentum, rather than a statics problem. It's convenient to use τ as shorthand for moment:

$$\tau = F d_{\perp}$$

The unit of torque[1] or moment is the Newton-meter : $[\tau] = \mathrm{N\,m}$
The critical part of working out the moment of a force is that the displacement from the axis of rotation and the force are perpendicular. This means that if you have an object like 4.1 then you will need to resolve *either* the force *or* the displacement. Which one you choose resolve is immaterial, so you might as well use whichever is most convenient!

[1] You'll not learn about angular momentum unless you take the sports science option

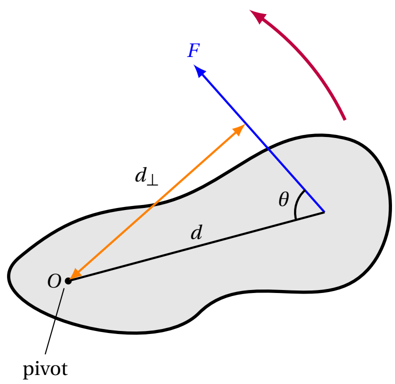


Figure 4.1: Moment of a shape

Example 4.1 In the diagram, if we resolve the displacement;

$$d_{\perp} = d \sin\theta$$

the moment of this force is:

$$\tau = F d \sin\theta$$

If we resolve the force:

$$F_{\perp} = F \sin\theta$$

so moment of this force is:

$$\tau = F \sin\theta d$$

The moment of a force is treated[2] [3] it can act in *clockwise* or *anti-clockwise* direction.
The moment of a force produces turning effects,[4] if there exists a non-zero moment, the object will have clockwise or anti-clockwise *angular acceleration*.

– Note that moment of a force depends on choice of pivot location. The moment of the same force with respect to different points can be very different. We often use this trick in questions to reduce the difficulty of the problem or the number of calculations required.

Torque of couple

Let's take a pair of equal but opposite forces acting at different positions on the same object, as in the figure 4.1.
The two forces produce moments in the same direction, but because they have equal magnitudes and opposite directions, they produce no translational acceleration. The combined effect is called the torque of a couple. The resultant torque due to the couple is:

$$\tau = Fd_{\perp,1} + Fd_{\perp,2} = 2F(d_{\perp,1} + d_{\perp,2})$$

$d_{\perp,1} + d_{\perp,2}$ is perpendicular distance $L_\perp$ between the pair, so:

$$\tau = FL_\perp$$

Torque of a couple can be therefore defined as the product of one force in the couple and the perpendicular distance between the pair

Note that: the torque of couple does not depend on choice of pivot. For same force pair, resultant moment is constant about any point.

Moment of weight

If you recall that weight is a force of gravity which is actually experienced by *all* parts of the object, then you will realise we need to sum up torques on each part of this object. This this brings a problem since the displacement of every part of the object will be different, fortunately, this calculation can be simplified using the idea of centre of gravity.

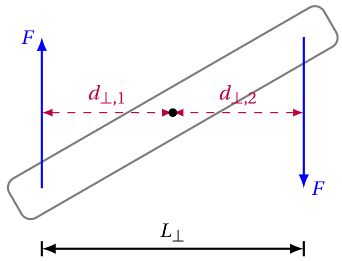
centre of gravity is a point at which the entire weight of an object is considered to act

There is a similar concept called the centre of mass;
centre of mass is the average position of all the mass that makes up the object.
Near the surface of the earth, mass and weight are directly proportional to each other so centre of mass is interchangeable with centre of gravity if we stay on earth[5].

[2] ⚠Rigorously speaking, moment is a *pseudovector*, which means that it does not transform quite like a normal vector although it does have a direction. In particular, if an object acted by a force is reflected across a plane, the moment of this force would not be reflected. Instead, it would be reflected and *reversed* as a vector quantity

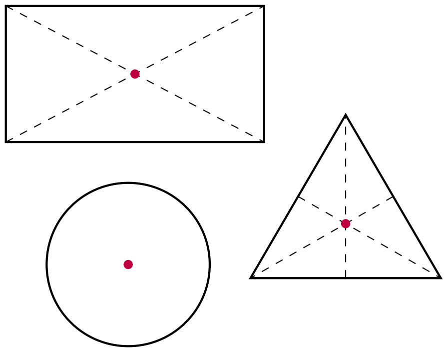
[3] ⚠Using vector notation, moment of a force can be defined as a *cross product*: $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector from the pivot to the point at which the force is applied.

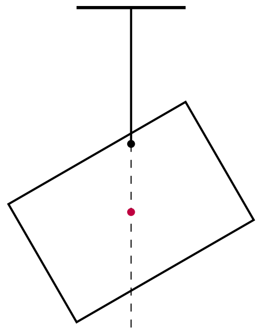
[4] Moment is like the rotational counterpart of a force: force changes the state of translational motion, moment changes the state of rotation.



[5] ⚠You might like to explore this idea a bit - when and where would they be in a different part of a body?

For a regularly-shaped uniform object, the centre of gravity/mass is its geometrical centre, If an object is hung freely, centre of gravity/mass is vertically below the point of suspension, otherwise weight would produce a non-zero torque about the point of suspension, causing the object to rotate until torque becomes zero:





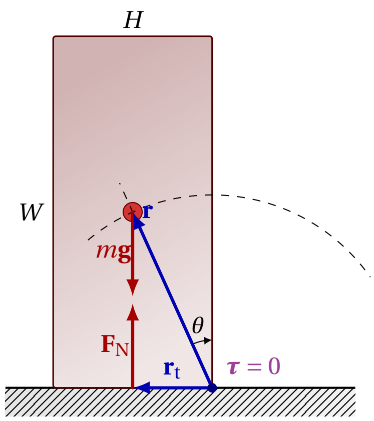
centre of mass of uniform lamina is at the geometrical centre

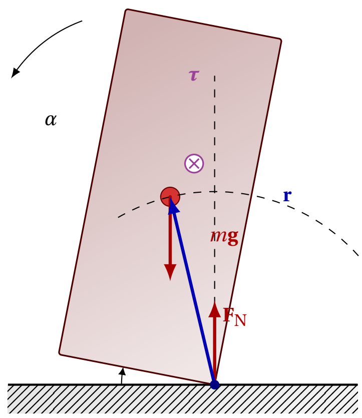
centre of mass is vertically below the point of suspension

A simple, practical way to find the centre of gravity/mass of an object is to suspended it from several positions. Each time we draw a vertical *plumb-line* through the point of suspension. The centre of mass/gravity lies where the lines intersect.

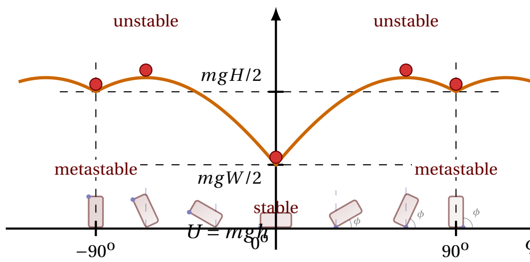
Stability

By using the idea of torque, we can easily understand how, or why some items are stable, and others less-so. When an object is sat on a level surface, undisturbed, the resultnt forces must pass through the centre of mass, resulting in no torque:





As soon as the block is disturbed from this position, there will be a resultant torque. If the torque produced causes the block to return to it's origional position we would say the object is *stable*.
As you can see - there can be more than one stable position for most shapes;
[6]:



[6] except the Gombok, which is only stable in one orientation

If we apply these ideas to the real world we can use them to understand that one of the more impressive looking tricks of the circus can be brought into the hands of everyone. Consider a tightrope walker:

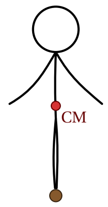
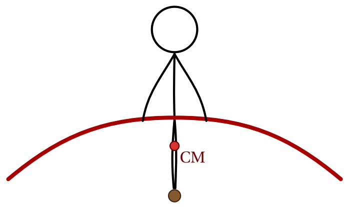


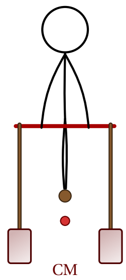
Figure 4.2: Walking on a tightrope unaided.

We can see that the Centre of Mass is substantially higher than the contact point and the situation is unstable. The artist has to be quite active in maintaining their CoM above the contact point - it's hard work!

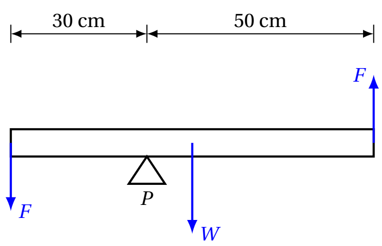
One of the tricks that you can use however is a long pole to assist balance. Not only does this give you a mass that's easy to move around but it lowers the center of mass:



This might be clearer if we picture the kind of balance support given to beginners:



Example 4.2 The diagram shows a uniform beam of weight $W = 20$ N and length 80 cm pivoted at point P. P is 30 cm from one end. Two equal but opposite forces of magnitude $F = 12$ N are acting at the two ends of the beam as shown. What is the resultant moment about point P?



> Moment of weight: $\tau_w = W d_w = 20 \times \left(0.50 - \frac{1}{2} \times 0.80\right) = 2.0$ N m (clockwise)
> Torque of couple: $\tau_c = FL = 12 \times 0.80 = 9.6$ N m (anti-clockwise)
> Resultant moment: $\tau_{\text{net}} = \tau_c - \tau_W = 9.6 - 2.0 = 7.6$ N m (anti-clockwise)

Principle of moments

If there is no turning effect for an object, the total moment of all forces must vanish:

> For a rigid body in equilibrium, sum of all clockwise moments must be equal to the sum of anti-clockwise moments *about any point*, this is called the **principle of moments**

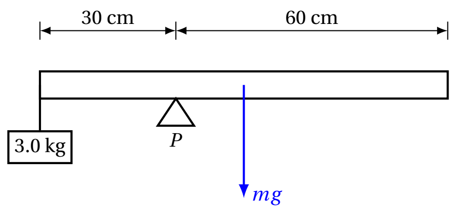
IE.an object in equilibrium has no turning effect about any point so resultant moment is zero about any point.
7

[7] As long as there is no resultant force, then zero resultant moment about any particular point would imply zero resultant moment about any point. Mathematically, let's take a collection of forces $\vec{F_1}, \vec{F_2}, \cdots, \vec{F_n}$ acting at positions $\vec{r_1}, \vec{r_2}, \cdots, \vec{r_n}$ on an object with respect to some fixed point O. Suppose their resultant moment vanishes, i.e., $\sum \vec{\tau_i} \equiv \sum \vec{r_i} \times \vec{F_i} = 0$, and also their resultant force vanishes, i.e., $\sum \vec{F_i} = 0$. If we focus on a different point O' with a relative displacement $\vec{R}$ to point O', then taking moments about O', we will have:

$$\sum \vec{\tau'_i} = \sum \vec{r'_i} \times \vec{F_i} = \sum (\vec{r_i} + \vec{R}) \times \vec{F_i} = \sum \vec{r_i} \times \vec{F_i} + \vec{R} \times \sum \vec{F_i} = 0$$

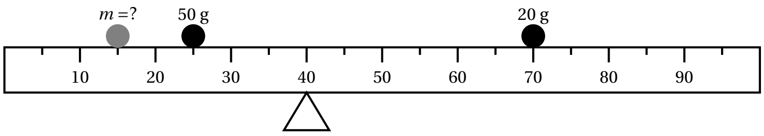
which shows zero resultant moment about one point together with zero resultant force guarantee resultant moment must be zero about any point in space.

Example 4.3 A uniform rod of length 90 cm is pivoted 30 cm from one end. It is balanced with a 3.0 kg load. Find the mass of the rod.



> Take moments about P:
> $3.0 \times 9.81 \times 0.30 = m \times 9.81 \times \left(0.60 - \frac{1}{2} \times 0.90\right)$
> so we find mass of rod: $m = 6.0$ kg

Example 4.4 A student balances a metre rule of mass 120 g supported on a fulcrum at the 40 cm mark. She then places a 20 g mass on the 70 cm mark and a 50 g mass on the 25 cm mark as shown. To balance the rule, what mass should she place on the 15 cm mark?

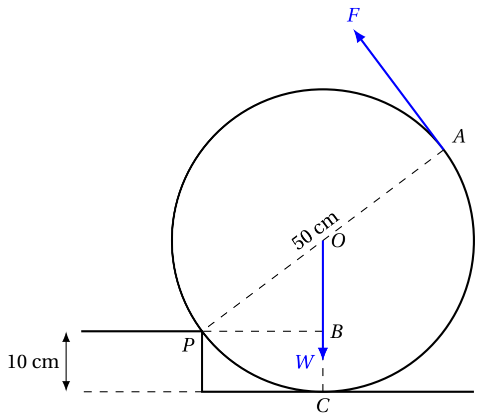


Take moments about the support:

$$mg \times (40 - 15) + 0.050g \times (40 - 25) =$$

$$0.12g \times (50 - 40) + 0.020g \times (70 - 40) \quad \Rightarrow \quad m = 42 \text{ g}$$

Example 4.5 A cylinder of weight 100 N and diameter 50 cm rests against point P of a curb of height 10 cm. What is the minimum force required to cause the cylinder to roll to the left?



To roll the cylinder, force applied must produce a torque no less than that of weight. The force is minimum if perpendicular displacement is greatest. Take moments about P (see diagram):
$F_{\text{min}} \times |PA| = W \times |PB|$
Note that $|PB| = \sqrt{|OP|^2 - |OB|^2}$, so $|PB| = \sqrt{0.25^2 - (0.25 - 0.10)^2} = 0.20$ m
Plug back into the equation above:

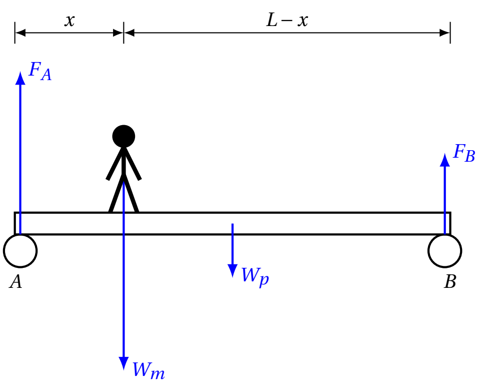
$$F_{\text{min}} \times 0.50 = 100 \times 0.20 \quad \Rightarrow \quad F_{\text{min}} = 40 \text{ N}$$

Mechanical equilibrium

Combining Newton's first law and principle of moments, we have the following statement:

For any mechanical system in equilibrium, two conditions must be satisfied:
- resultant force is zero in any direction: $\sum F = 0$
- resultant moment is zero about any point: $\sum \tau = 0$

These two conditions allow for many problems to be solved.
Example 4.6 A uniform plank of weight 100 N and length $L = 6.0$ m rests horizontally on two supports A and B. A man of weight 800 N stands a distance of $x = 1.5$ m from end A. Determine the forces acting at the two supports.



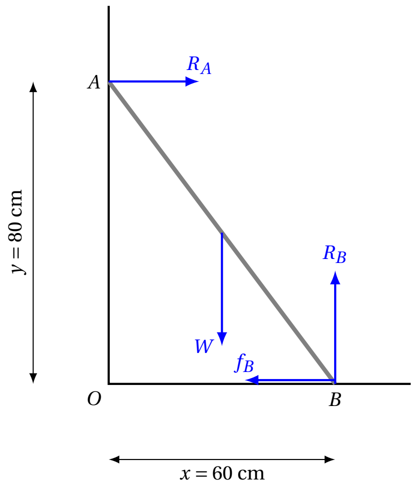
Take moments about A: $W_m x + W_p \cdot \frac{L}{2} = F_B L$

$$800 \times 1.5 + 100 \times 3.0 = F_B \times 6.0 \Rightarrow F_B = 250\text{ N}$$

Take moments about B: $W_m(L-x) + W_p \frac{L}{2} = F_A L$

$$800 \times 4.5 + 100 \times 3.0 = F_A \times 6.0 \Rightarrow F_B = 650\text{ N}$$

One can check that: $F_A + F_B = W_m + W_p$, there must be no resultant force in vertical direction



Example 4.7 A uniform ladder of weight 120 N rests on a rough ground against a smooth wall as shown. The dimensions are labelled on the diagram. (a) What is the contact force acting at B? (b) What is the contact force acting at A? (c) What is the frictional force at B?

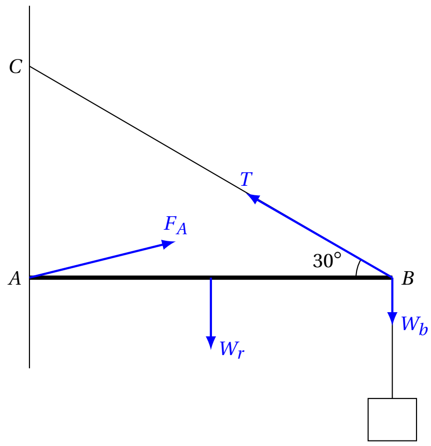
Free-body diagram is drawn as shown

resolve vertically: $R_B = W \quad \Rightarrow \quad R_B = 120\text{ N}$

take moments about B: $R_A y = W\frac{x}{2}$

$$R_A = \frac{120 \times 0.30}{0.80} = 45\text{ N}$$

resolve horizontally: $f_B = R_A \quad \Rightarrow \quad f_B = 45\text{ N}$



Example 4.8 The diagram shows a uniform rod AB of weight 60 N being held horizontally to a vertical wall by means of a light string. The string is attached to the rod at B, where a basket of weight 40 N is suspended. The other end of the string is fixed on the wall at C. The angle between the string and the rod is 30°. (a) Find the tension in the string. (b) Find the force acting on the rod at point A.

Take moments about A: $TL\sin\theta = W_b L + W_r \frac{1}{2} L$

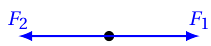
$$T\sin 30° = 40 + 60 \times \frac{1}{2} \quad \Rightarrow \quad T = 140\text{ N}$$

resolve horizontally: $F_{A,x} = T\cos\theta \quad \Rightarrow \quad F_{A_x} = 140\cos 30° \approx 121\text{ N}$

resolve vertically: $F_{A,y} + T\sin\theta = W_r + W_b \quad \Rightarrow \quad F_{A,y} = 60 + 40 - 140\sin 30° = 30\text{ N}$

force at A: $F_A = \sqrt{F_{A,x}^2 + F_{A,y}^2} \quad \Rightarrow \quad F_A = \sqrt{121^2 + 30^2} \approx 125\text{ N}$

Two forces in equilibrium



The problem of two balanced forces is trivial

Suppose two forces F_1 and F_2 are acting on an object in equilibrium

- to have zero resultant force, F_1 and F_2 must be equal but opposite
- to have zero resultant moment, F_1 and F_2 must act along same line
  otherwise they produce torque of couple that causes turning effects

Three forces in equilibrium

Force triangle

When there are more than two forces, the situation becomes more complicated and we use *vector diagrams* to solve the problems.
Suppose a set of forces $\overrightarrow{F_1}, \overrightarrow{F_2}, \cdots, \overrightarrow{F_n}$ are in equilibrium. If that's the case then no resultant force requires $\overrightarrow{F_1} + \overrightarrow{F_2} + \cdots + \overrightarrow{F_n} = 0$
Recall that resultant force is vector sum of all forces acting, and now this sum has to vanish, so if the force vectors are connected head to tail, they should form a closed n-polygon:

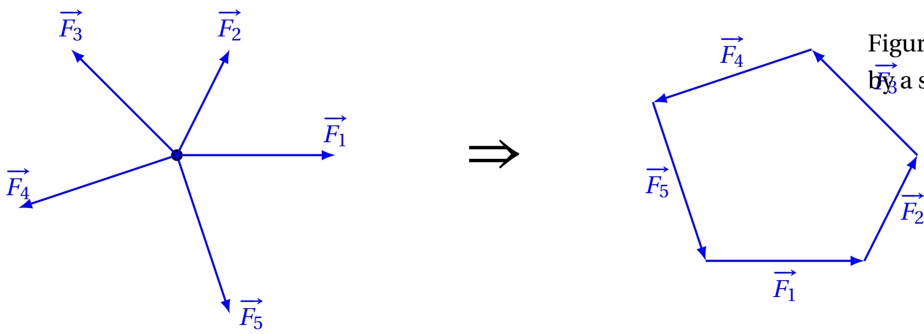
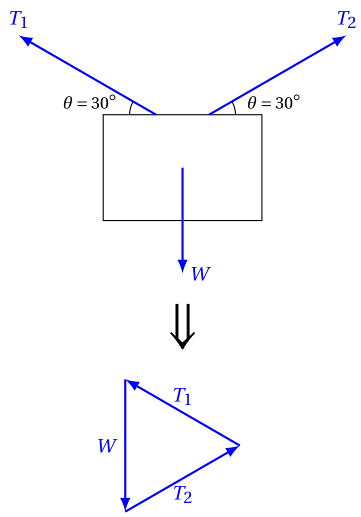


Figure 4.1: an n-polygon formed by a set of n balanced forces

In the case of three balanced forces, net force is zero means they should form a **force triangle**. Any unknown forces can then be solved by cracking a geometric problem.

Example 4.9 A painting of weight $W = 20$ N is supported by two strings as shown. Both strings form an angle $\theta = 30°$ to the horizontal. Find the tension in the strings.



By resolving forces, we have:

$$\begin{cases} T_1 \cos\theta = T_2 \cos\theta \\ T_1 \sin\theta + T_2 \sin\theta = W \end{cases}$$

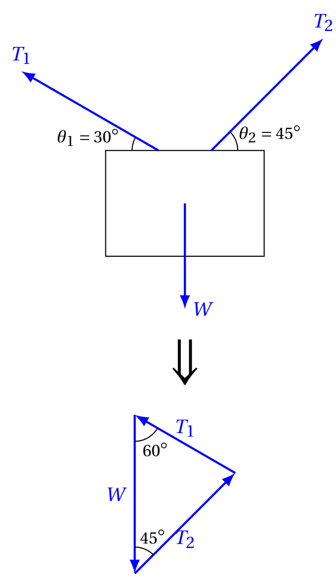
we solve the equations to obtain:

$$T_1 = T_2 = \frac{W}{2\sin\theta} = \frac{20}{2\sin 30°} = 20\text{ N}$$

alternatively, we can construct the force triangle as shown
T_1, T_2 and W form an equilateral triangle, so

$$T_1 = T_2 = W = 20\text{ N}$$

Example 4.10 The same painting of weight $W = 20$ N is supported by two strings at different angles $\theta_1 = 30°$ and $\theta_2 = 45°$ as shown. Find the forces in the two strings.



By resolving forces, we have:

$$\begin{cases} T_1 \cos\theta_1 = T_2 \cos\theta_2 \\ T_1 \sin\theta_1 + T_2 \sin\theta_2 = W \end{cases} \Rightarrow \begin{cases} \frac{\sqrt{3}}{2} T_1 = \frac{\sqrt{2}}{2} T_2 \\ \frac{1}{2} T_1 + \frac{\sqrt{2}}{2} T_2 = 20 \end{cases}$$

solving this, we find: $T_1 \approx 14.6\,\text{N}$, $T_2 \approx 17.9\,\text{N}$

alternatively, we construct the force triangle as shown

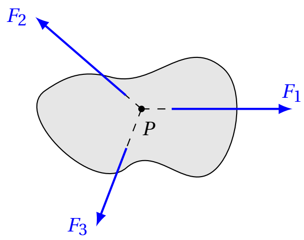
T_1 and T_2 are related to W by the the law of sine:

$$\frac{W}{\sin 75°} = \frac{T_1}{\sin 45°} = \frac{T_2}{\sin 60°}$$

from this we get the same result: $T_1 \approx 14.6\,\text{N}$, $T_2 \approx 17.9\,\text{N}$

Concurrent forces

Tor three forces in equilibrium, they must produce zero resultant moment about any point.
Suppose lines of action of F_1 and F_2 meet at point P, the moment of F_1 and moment of F_2 about P are both zero. To produce zero resultant moment about P, then moment of F_3 about P must vanish which suggests that the line of action of F_3 must pass through point P. The lines of action of F_1, F_2 and F_3 therefore, must pass through the same point[8]
Such three forces are said to be *concurrent*



[8] In the case of three parallel forces in equilibrium, we can introduce the notion of an ideal point at infinity, so that parallel lines could meet at that point(!).

Summary for three forces in equilibrium

For three forces in equilibrium, we can now conclude:

- the three force vectors must be able to form a force triangle
  this is a consequence of zero resultant force
- the lines of action for the three forces must pass through same point
  this is a consequence of zero resultant moment

End-of-chapter questions

Mechanical equilibrium

Question 4.1 Is it possible for an object to be in equilibrium if only one force is acting on it?
Question 4.2 If three forces are in equilibrium, suggest and explain whether the lines of action must lie in the same plane.

5 Momentum

Momentum

momentum of an object is defined as the product of its mass and its velocity:

$$p = mv$$

Momentum is a vector quantity, in the same direction as the object's velocity. It has units of $[p] = \text{kg m s}^{-1} = \text{N s}$
To find change in momentum of a body, or to find sum of the momenta[1] for a system of several objects, one has to keep track of directions.

[1] Momenta is the plural form of momentum.

Momentum and force

Suppose a constant net force F is applied on a body, we can write:

$$F = ma = m\frac{\Delta v}{\Delta t} = \frac{m\Delta v}{\Delta t} = \frac{\Delta p}{\Delta t}$$

This is closer to Newton's origional statement that we have reduced to mass times acceleration at GCSE.
We can give a formal definition for Force:

Force is defined as an effect that causes a rate of change in momentum:

$$F = \frac{\Delta p}{\Delta t}$$

Information about *change* in momentum can be deduce from force-time graphs

area under F-t graph equals the change in object's momentum

From this definition, we see that the force acting depends on the magnitude of the change in momentum, but also depends on how long this change occurs. It would be silly to land on the ground with stiff legs. You naturally bend your knees. Acrobats in a circus fall on a soft mat or a safety net. All these actions does not change the impulse, but they increase the time of contact, and therefore reduce the force that might cause harmful injuries.

Example 5.1 A ball of 120 g strikes a wall at right angle with a speed of 10 m s^{-1}. It rebounds with the same speed. If the time of impact is 25 ms, find the average force exerted on the ball.

change in momentum: $\Delta p = mv - mu = 0.12 \times 10 - 0.12 \times (-10) = 2.4\ \text{kg m s}^{-1}$
average force: $F = \frac{\Delta p}{\Delta t} = \frac{2.4}{2.5 \times 10^{-3}} = 96\ \text{N}$

Example 5.2 Water is pumped through a hose-pipe. A man is holding the hose-pipe horizontally and water emerges from the hose-pipe with a speed of 16 m s^{-1} at a rate of 45 kg per minute. Find the force required from this man to hold steady the hose-pipe.

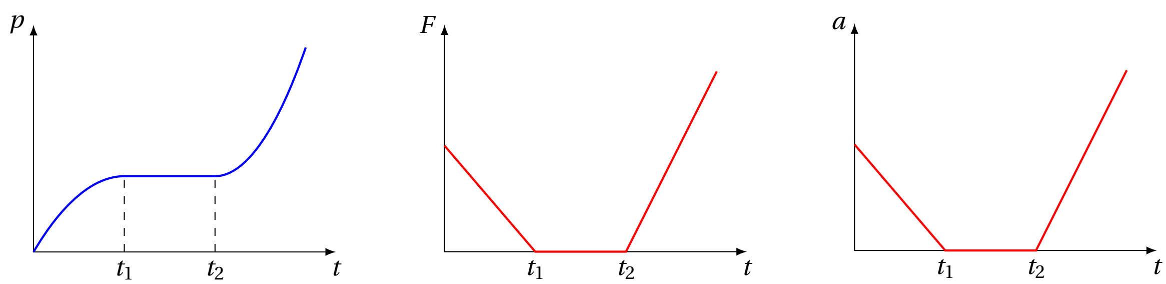
$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{\Delta m(v-0)}{\Delta t} = \frac{45 \times 16}{60} \quad \Rightarrow \quad F = 12\ \text{N}$$

Example 5.3 A strong wind of speed 30 m s^{-1} blows against a wall of area 10 m^2 at right angles. The density of the air is 1.2 kg m^{-3}. Assume air speed reduces to zero when it hits the wall. What is the approximate force exerted by the air on the wall?

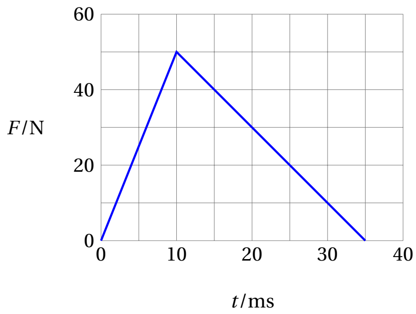
$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{\Delta m(v-0)}{\Delta t} \overset{m=\rho V}{=\!=\!=} \frac{\rho \Delta V v}{\Delta t} \overset{V=AL}{=\!=\!=} \frac{\rho A \Delta L v}{\Delta t} \overset{\Delta L = v\Delta t}{=\!=\!=} \rho A v^2$$

$$\Rightarrow \quad F = 1.2 \times 10 \times 30^2 = 10800\ \text{N}$$

Example 5.4 Given the variation with time of the momentum of a body as shown in the p-t graph, check yourself that the variation of force acting and the variation of the object's acceleration should be plotted as shown in the F-t graph and the a-t graph.



Example 5.5 An object of mass 70 g is initially at rest. A force that varies with time is exerted on the object. The graph shows the how the force varies during the time of impact. What is the final velocity of the object?



area under F-t graph gives change in momentum

$$\Delta p = \frac{1}{2} \times 50 \times 35 \times 10^{-3} = 0.875\ \mathrm{kg\ m\ s^{-1}}$$

final velocity: $v = \frac{\Delta p}{m} = \frac{0.875}{70\times 10^{-3}} = 12.5\ \mathrm{m\ s^{-1}}$

Impulse-momentum relation

We can further introduce a quantity called the impulse. **Impulse** is defined as the product of a force and time interval at which it acts: $J = F\Delta t$

$F\Delta t$ gives change in momentum, therefore we have the impulse-momentum relation: $J = \Delta p$

impulse-momentum relation is just another way of expressing Newton's second law

but the idea of impulse and momentum allows analysis of variable mass and information about force can be extracted from momentum-time graphs.

The gradient of p-t graph equals the resultant force acting

Principle of momentum conservation

The change in an object's momentum is given by: $\Delta p = F\Delta t$ [2] in particular, if there is zero net force, then object's momentum stays constant. This idea can be generalised to a system of objects;

There are external forces from outside and internal forces between objects within the system - let's take two mutually interacting objects within the system, say A and B. By Newton's 3rd law, force on A by B is equal and opposite to force on B by A. The change in A's momentum by B is then equal and opposite to change in B's momentum by A.

The change in total momentum of A and B due to each other is therefore cancelled out. For the system as a whole, effect of internal forces always cancel out so the change of total momentum of the system would only depend on net external force. [3]

If there is no net external force, then there is no change in total momentum.

for any closed system where net external force is zero, the total momentum of the system remains constant, this is called the **principle of momentum conservation**

Example 5.6 A uranium-238 nucleus disintegrates, emitting an α-particle of mass 4u and producing a thorium-234 nucleus of mass 234u.

[2] The relation $\Delta p = F\Delta t$ is valid if we are dealing with a *constant* force. If the object is acted by a varying force, then the change in its momentum is given by: $\Delta p = \int F \mathrm{d}t$.

[3] A more rigorous derivation goes as follows.
Let's consider a system of point objects m_i, each experiences a resultant force F_i where F_i can come from some external source or another object j within the system: $F_i = F_{i,\text{ext}} + \sum_j F_{i,j}$.
Summing over all objects, we can write: $\sum_i F_i = \sum_i F_{i,\text{ext}} + \sum_{i,j} F_{i,j}$
For each pair i and j, the action-reaction principle suggests that the mutual interaction between the two are equal but opposite: $F_{i,j} = -F_{j,i}$, so $\sum_{i,j} F_{i,j} = 0$.
Therefore, $\sum_i F_i = \sum_i F_{i,\text{ext}}$.
Multiply both sides by Δt, we can write: $\sum_i F_i \Delta t = \sum_i F_{i,\text{ext}} \Delta t$. Note that $\sum_i F_i \Delta t = \sum_i \Delta p_i$ gives the change in total momentum of system, so this shows the change of total momentum is determined by the net external force:

$$\left(\sum_i F_{i,\text{ext}}\right)\Delta t = \sum_i \Delta p_i$$

The uranium nucleus is initially at rest. (a) What is the ratio of the velocities of the product particles $\frac{v_\alpha}{v_{\text{Th}}}$? (b) Explain why the α-particle and the thorium nucleus must be emitted in opposite directions.

> No external force involved during decay, so momentum is conserved. Zero initial momentum means total momentum of α-particle and thorium nucleus is zero
> α-particle and thorium nucleus must carry equal but opposite momenta
> equal momentum $\Rightarrow \quad m_\alpha v_\alpha = m_{\text{Th}} v_{\text{Th}} \quad \Rightarrow \quad \frac{v_\alpha}{v_{\text{Th}}} = \frac{m_{\text{Th}}}{m_\alpha} = \frac{234}{4} = 58.5$
> They have opposite momentum so they move off in exactly opposite directions

Collision problems

For two bodies colliding together, external forces are negligible during the time of contact, hence total momentum is considered to be conserved for any collision process.

Collision in one dimension

Suppose two masses m_1 and m_2 are restricted to move in one dimension only, they move at initial velocities u_1 and u_2 before they collide. After collision, their velocities become v_1 and v_2, as depicted in the diagram.

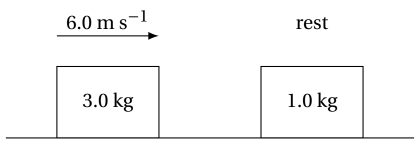
Before collision. After collision.

The total momentum is conserved, so:

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

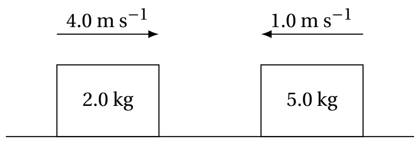
Momentum and velocity are both vector quantities, we normally choose objects moving to the right to have positive momentum/velocity, then object travelling to the left would have negative momentum/velocity.

Example 5.7 A 3.0 kg mass moving at 6.0 m s^{-1} has a head-on collision with a stationary 1.0 kg mass. The two masses stick together on impact. What is the final velocity of the two masses?



> $$m_1 u_1 + \cancel{m_2 u_2}^{\,0} = (m_1 + m_2) v \quad \Rightarrow \quad 3.0 \times 6.0 = (3.0 + 1.0) \times v$$
> $$\Rightarrow \quad v = 4.5 \text{ m s}^{-1}$$

Example 5.8 A 2.0 kg mass moving at 4.0 m s^{-1} collides head on with a 5.0 kg mass moving at 1.0 m s^{-1}. After the collision, speed of the 5.0 kg



mass is unchanged but its direction is reversed. What is the velocity of the 2.0 kg mass after the collision?

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$\Rightarrow 2.0 \times 4.0 + 5.0 \times (-1.0) = 2.0 \times v_1 + 5.0 \times 1.0 \Rightarrow v_1 = -1.0 \text{ m s}^{-1}$$

minus sign means the 2.0 kg mass reverses direction after collision

Elastic & Inelastic collisions

Elastic collisions

Collisions are usually treated as either an elastic or an inelastic interaction.[4]

An **elastic collision**, is an interaction in which there is no loss of kinetic energy.

We can derive a condition that must be satisfied by two objects colliding elastically. Since momentum and kinetic energy are both conserved, we can write two equations:[5]

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$
$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Rearranging both equations, we have:

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \qquad (1)$$
$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \qquad (2)$$

Equation (4) can be further rewritten as:

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2) \qquad (2')$$

Comparing equation (2') to equation (1), one has: $u_1 + v_1 = v_2 + u_2$
Rearrange the equation, we find: $v_2 - v_1 = u_1 - u_2$
Both side of the equation now represent a *relative speed* between the two colliding bodies.

For an elastic collision process between two bodies, the relative velocity of separation after collision equals the relative velocity of approach before collision.

[4] Here we assume you already have some knowledge about *kinetic energy*. Kinetic energy of a moving body is given by the formula: $E_k = \frac{1}{2} m v^2$. You might have learned about it in an GCSE course or elsewhere. We will talk about kinetic energy in §6.3.

[5] For simplicity, we consider two-body collision in one dimension only, that is the two bodies move along the same straight line before and after the collision. For a two-body collision problem in two dimension, the conservation of momentum can be broken into two independent component equations.

Inelastic collisions

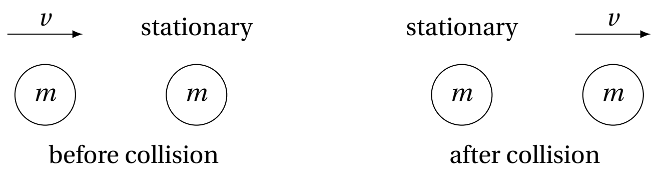
In an **inelastic collision**, part of kinetic energy is lost due to change in object's shape. ie we cannot use conservation of energy.

Brief summary

Discussions on elastic and inelastic collisions are summarised in the table below:

	elastic collision	inelastic collision
conservation of momentum	✓	✓
conservation of kinetic energy	✓	✗
conservation of total energy	✓	✓
relative speed stays unchanged	✓	✗

Example 5.9 A sphere of mass m moves on a smooth horizontal surface at speed v and collides *elastically* with an identical ball at rest. What are the final velocities of the two spheres?



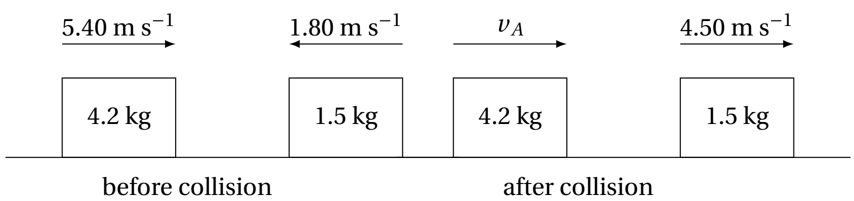
$$\begin{matrix} mv = mv_1 + mv_2 & \text{(momentum conservation)} \\ v = v_2 - v_1 & \text{(relative speed unchanged)} \end{matrix} \Rightarrow \begin{cases} v_1 = 0 \\ v_2 = v \end{cases}$$

The two spheres simply exchange velocities during the collision, as shown in diagram

[6]

[6] If two objects of equal mass collide elastically with one another, one can actually show that their velocities would exchange regardless of their initial velocities.

Example 5.10 A 4.2 kg mass A and a 1.5 kg mass B are travelling towards each other on a frictionless horizontal plane. Mass A and B move at 5.40 m s^{-1} and 1.80 m s^{-1} respectively before they strike, as shown below. Mass B moves to the right at 4.50 m s^{-1} after the collision, (a) find the velocity of A after the impact, and (b) suggest whether the collision is elastic.

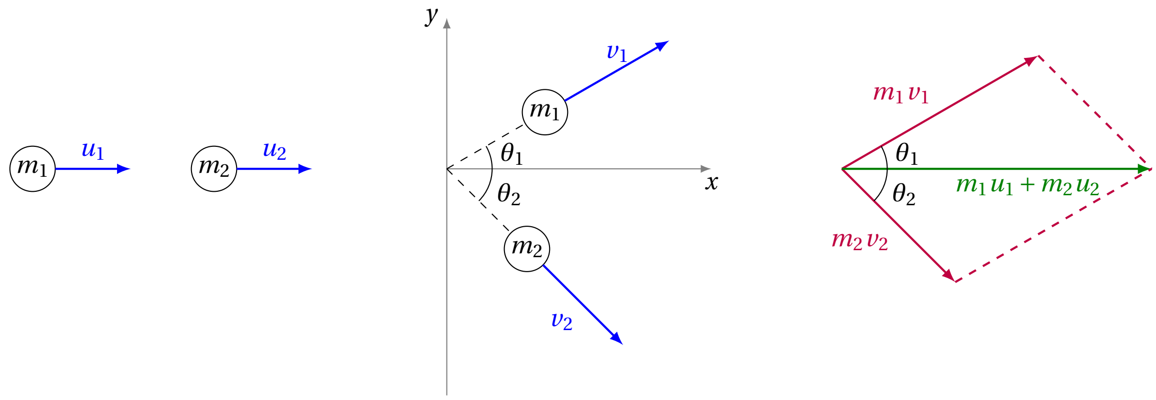


Momentum conserved: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$
$4.2 \times 5.40 + 1.5 \times (-1.80) = 4.2 \times v + 1.5 \times 4.50 \quad \Rightarrow \quad v_A = 3.15 \text{ m s}^{-1}$
K.E. before: $E_{k,i} = \frac{1}{2} m_A u_A^2 + \frac{1}{2} m_A u_B^2 = \frac{1}{2} \times 4.2 \times 5.40^2 + \frac{1}{2} \times 1.5 \times 1.80^2 \approx 63.7 \text{ J}$
K.E. after: $E_{k,f} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_A v_B^2 = \frac{1}{2} \times 4.2 \times 3.15^2 + \frac{1}{2} \times 1.5 \times 4.50^2 \approx 36.0 \text{ J}$
there is K.E. loss, so collision is inelastic
alternatively, we can compare the relative speed before and after the collision
$u_A - u_B = 5.40 - (-1.80) = 7.20 \text{ m s}^{-1} \qquad v_B - v_A = 4.50 - 3.15 = 1.35 \text{ m s}^{-1}$
relative speed changed after the collision, so collision must be inelastic

Collisions in two dimensions

When objects collide on a horizontal plane, they can possibly move off in any direction. If negligible net external force is present, total momentum is still conserved - recall that momentum is a vector quantity, so momentum should be conserved in any direction.

Let's consider the collision between two masses m_1 and m_2. For simplicity, assume their initial velocities u_1 and u_2 are in same direction. Final velocities v_1 and v_2 after the collision are shown



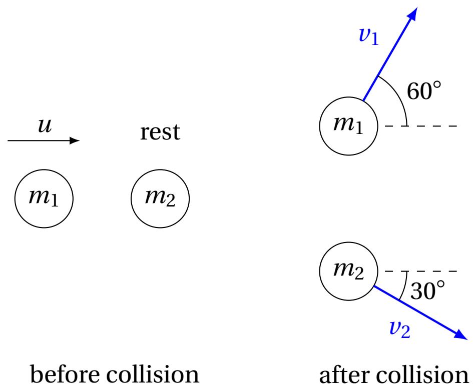
before collision | after collision | momentum conservation

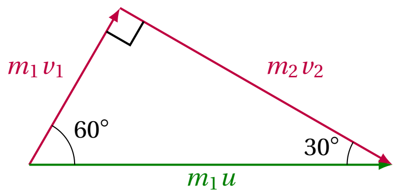
Equations of momentum conservation can be written for two perpendicular directions:

$$\begin{aligned} m_1 u_1 + m_2 u_2 &= m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 && \text{(in } x\text{-direction)} \\ 0 &= m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2 && \text{(in } y\text{-direction)} \end{aligned}$$

One can also construct a vector triangle to transform the problem into a geometry problem.

Example 5.11 A ball of mass $m_1 = 1.0$ kg travelling with a speed of $u = 6.0 \text{ m s}^{-1}$ in the x-direction strikes a stationary ball of mass $m_2 = 2.0$ kg.





The direction of the balls' velocities v_1 and v_2 after the collision are shown in the diagram. Find v_1 and v_2.

Start with momentum conservation equations:

$$\begin{cases} m_1 u = m_1 v_1 \cos 60^\circ + m_2 v_2 \cos 30^\circ \\ 0 = m_1 v_1 \sin 60^\circ - m_2 v_2 \sin 30^\circ \end{cases}$$

$$\begin{cases} 1.0 \times 6.0 = 1.0 \times v_1 \times \frac{1}{2} + 2.0 \times v_2 \times \frac{\sqrt{3}}{2} \\ 0 = 1.0 \times v_1 \times \frac{\sqrt{3}}{2} - 2.0 \times v_2 \times \frac{1}{2} \end{cases}$$

Simplify and solve the equations:

$$\begin{cases} \frac{1}{2} v_1 + \sqrt{3} v_2 = 6 \\ v_2 = \frac{\sqrt{3}}{2} v_1 \end{cases} \Rightarrow \begin{cases} v_1 = 3.0 \text{ m s}^{-1} \\ v_2 \approx 2.6 \text{ m s}^{-1} \end{cases}$$

One can also draw and use the vector triangle for this question, this happens to be a right-angled triangle, so things become much easier.

$$\begin{cases} m_1 v_1 = m_1 u \cos 60^\circ \\ m_2 v_2 = m_1 u \cos 30^\circ \end{cases} \Rightarrow \begin{cases} 1.0 \times v_1 = 1.0 \times 6.0 \times \frac{1}{2} \\ 2.0 \times v_2 = 1.0 \times 6.0 \times \frac{\sqrt{3}}{2} \end{cases} \Rightarrow$$

$$\begin{cases} v_1 = 3.0 \text{ m s}^{-1} \\ v_2 \approx 2.6 \text{ m s}^{-1} \end{cases}$$

End-of-chapter questions

Force & momentum

Question 5.1 When speed cars run out of control in a racing game, why are they stopped by haystacks instead of concrete walls? When you jump from an elevated position and land on the ground, you naturally bend your knees instead of keeping your legs stiff. How does that reduce the chance of causing harmful injuries?

Question 5.2 Automobiles were manufactured to be as rigid as possible, but nowadays many cars are designed to crumple upon impact. Can you explain why?

Principle of momentum conservation

Question 5.3 How does conservation of momentum apply to a ball bouncing off a wall?
Question 5.4 In the comic hero series, Superman hurls an asteroid in outer space, and he is seen at rest after the throw. What law of physics is violated here? If the asteroid is 100 times as massive as the superhero, and it is thrown at 50 m s^{-1}. What is Superman's velocity right after the throw?

Collision problems

Question 5.5 When a piece of putty falls and hits the floor without bouncing, what becomes of its momentum before impact? What becomes of its kinetic energy?

6 *Work & Energy*

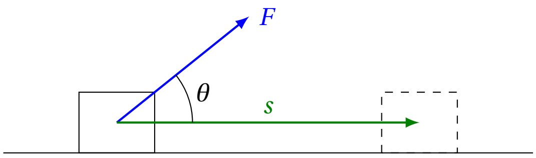
In the previous chapter, we studied the accumulative effect of a force over a period of time and have seen how this gives rise to the idea of impulse and momentum. In this chapter, we consider the effect of a force over a certain displacement, and you will learn how this is related to the concept of work done and energy.
Energy is a concept central to all of physical sciences. The entire universe is made up by energy and matter. In this section we study the energy changes during various physical processes.

Work

> **Work done** by a force is defined as the product of the force and the displacement moved out in the direction of the force:
>
> $$W = Fs$$

The unit for work done: is the Joule: $[W] = [F][s] = \mathrm{N \cdot m} \equiv \mathrm{J}$ (joule)
As with moments, the direction of the displacement or the force is important. Only the components that align contribute.If the force acts at angle θ to the displacement travelled, then:

$$W = Fs\cos\theta$$



Like any energy concept, work is a *scalar* quantity[1], i.e., it has no direction. Note that work done can be either positive or negative, this tells you the direction of energy transfer. If the work being done on an object is positive, then work is being done on it.
Resistive forces, such as friction and air drag, act in the opposite direction to motion so work done by resistive forces is negative[2], we say this is work done *against* resistance. The minus sign will be crucial in energy calculations in later discussions.

[1] Although work is defined as the product of two vectors, work carries no information about direction. This vector product is called as a *scalar product* or a *dot product*, which can be written explicitly as: $W = \vec{F} \cdot \vec{s} = |F||s|\cos\theta$. You might have seen this operation in the A-Level course in Mathematics.

[2] You might take $\theta = 180°$, then $\cos\theta = -1$, giving rise to a negative work done.

It is particularly common to worry about the direction of energy transfer when we are considering the behaviour of gasses because a change in energy is usually connected with a change in Temperature, and/or volume, and/or pressure.
If pressure stays constant, then work by/on gas: $W = F\Delta s = pA\Delta s \quad \Rightarrow$

$$W_{\text{gas}} = p\Delta V$$

Work done by a varying force[3] is found by integration or using a F-s graph. If force varies with position, its change over small displacements is still considered small. Work done over an infinitesimal displacement is therefore $\mathrm{d}W = F\mathrm{d}s$. Integrate from initial position to final position, total work done is given by:

[3] The equation $W = Fs$ is valid only if the force acting is constant. Similarly, the equation $W_{\text{gas}} = p\Delta V$ holds for constant pressure processes only.

$$W = \int_i^f F\mathrm{d}s$$

If one plots force against displacement, then area under F-s curve gives work done

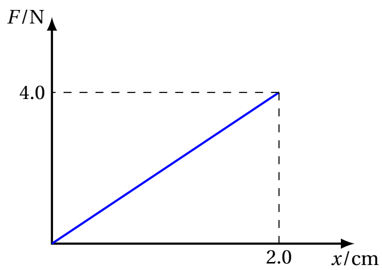
Example 6.1 A 20 N force is applied at 60° to the horizontal to move a 1.0 kg object at a constant speed of 2.0 m s^{-1} for 30 s. How much work is done by the force?

$$W = Fs\cos\theta = Fvt\cos\theta = 20 \times 2.0 \times 30 \times \cos 60^\circ \quad \Rightarrow \quad W = 600\ \text{J}$$

Example 6.2 A piston in a gas pump has an area of 600 cm^2. During one stroke, the pump moves a distance of 30 cm against a constant pressure of 8000 Pa. How much work is done?

$$W = p\Delta V = pA\Delta s = 8000 \times 600 \times 10^{-4} \times 30 \times 10^{-2} \quad \Rightarrow \quad W = 144\ \text{J}$$

Example 6.3 When a spring is compressed by 2.0 cm, the force applied increases uniformly from zero to 4.0 N. How much work is done by this force?



F-x graph for the force is plotted as shown
work to compress spring equals area under F-x graph: $W = \frac{1}{2} \times 4.0 \times 2.0 \times 10^{-2} = 0.040\ \text{J}$

Energy forms

Energy is not so much a thing as a useful abstract concept - there's a common misconception that there are "types" of energy that one can transform from one "type" to another. We do use this language sometimes for the sake of linguistic ease but it's important to understand that we're just using a useful shorthand that makes it easier to talk about, and predict how the universe behaves. We say that Energy is something acquired by an object that enables it to do work. A moving vehicle, water stored in a reservoir, a compressed spring, separated magnets, all of these objects can do work to other objects so we say they have a quantity of energy stored in some form. In this section, we will look at various situations where work on an object causes a change in some form of energy.

Kinetic energy

Suppose a constant force F is acting over a distance s, we have:

$$W = Fs \overset{F=ma}{=\!=\!=} mas \overset{v^2=u^2+2as}{=\!=\!=\!=\!=} m\frac{v^2-u^2}{2} \quad \Rightarrow \quad W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

This shows work is done and energy is transferred into some quantity associated with object's motion.
This is recognised as the gain in (the store of[4]) kinetic energy of the object: $\Delta E_k = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

[4] Whilst important at GCSE, we're past the point where we have to be obsessive about using the language of stores. We know energy is a weird abstract concept and that's ok

Kinetic energy (K.E.) is the energy stored by an object due to its motion

An object of mass m moving with speed v has K.E.:

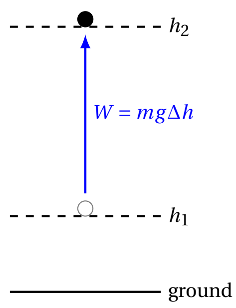
$$E_k = \frac{1}{2}mv^2$$

Example 6.4 Estimate the kinetic energy of a running man.

Suppose the man has a mass of 75 kg and is running at 5 m s^{-1} (any reasonable value will do)

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2} \times 75 \times 5^2 \approx 940\text{ J}$$

Gravitational potential energy



Later on we will consider situations that are in a nun-uniform gravitational field ie. on a planetary scale. For now, lets stick with human-scale things and so the value for "g" is constant everywhere. Consider a body being slowly pulled from a height of h_1 to h_2. The work done for this process is: $W = Fs = mg\Delta h = mgh_2 - mgh_1$
This shows work is transformed into change in some quantity associated with object's position, we say this is the gain in gravitational potential energy:

$$\Delta E_p = mgh_2 - mgh_1$$

Gravitational potential energy (G.P.E.) is the energy possessed by a body due to its position in a gravitational field

In a uniform field, a body of mass m at a height of h has G.P.E.:

$$E_p = mgh$$

G.P.E. is a *relative* quantity, only its change is important in physical processes [5]

Elastic potential energy

Let's now consider a spring being stretched or compressed, work is done by external forces to cause the change in shape - this is the mechanism that stores the elastic energy potential in the body.

> **Elastic potential energy**, also called **strain energy**, is the energy possessed by an elastic body due to deformation

This topic will be revisited in details in §7.2

Other energy stores

Apart from those we have mentioned above, there are many other ways to store energy:

– *electric potential energy*: energy of a charged object due to its position in an electric field.

– *chemical energy*: ability to do work due to potential energy between atoms and molecules.

– *nuclear energy*: ability to do work due to potential energy of sub-atomic particles in the nuclei.

– *internal energy*: sum of random kinetic and potential energies of molecules in a substance.

– *electromagnetic energy*: energy carried by light/electromagnetic waves.

Work & energy transformations

From previous discussions, we have seen doing work is a way of transferring energy, for examination purposes[6], we can say the following:

> The change in the total energy of an object equals the net work done by all external forces (excluding those associated with potential energies):
>
> $$W = \Delta E$$

Example 6.5 A racing car of 800 kg starts off from rest. If the driving force is 5000 N, and the car experiences a constant resistive force of 1500 N, what is its speed after it has travelled 50 m?

[5] The formula $E_p = mgh$ implies that we have defined G.P.E. at the ground level to be zero. But this is purely conventional. Here I would like to point out that one can freely choose any zero potential energy level as he/she wishes, but no matter what point is picked as reference, we will always agree on the quantity of physical significance, that is, the *change* in G.P.E between two fixed points.

[6] More rigorous discussions (which go beyond the syllabus) are given in §6.6.

The gain in K.E. equals work by driving force plus *negative* work against resistance

$$W_{\text{total}} = \Delta E_k \quad \Rightarrow \quad Fs - fs = \frac{1}{2}mv^2 - 0$$

$$(5000 - 1500) \times 50 = \frac{1}{2} \times 800 \times v^2 \quad \Rightarrow \quad v \approx 20.9 \text{ m s}^{-1}$$

Example 6.6 A concrete cube of side 0.50 m and density 2400 kg m^{-3} is lifted 4.0 m by a crane. How much work is done?

The work done by crane equals gain in G.P.E. of cube

$$W = \Delta E_p$$

$$\Rightarrow \quad W = mg\Delta h = \rho V g \Delta h = 2400 \times 0.50^3 \times 9.81 \times 4.0 \approx 1.18 \times 10^4 \text{ J}$$

conservation of energy

The law of conservation of energy states that energy cannot be created or destroyed, but can only transform from one form into another while the total amount is always constant.

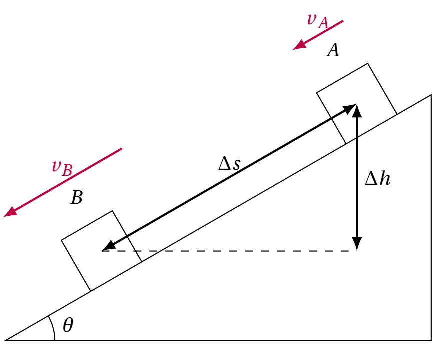
Example 6.7 For an object falling from rest due to gravity, if air resistance is negligible, what is its speed when it has fallen through a distance of h?

$$\text{G.P.E loss} = \text{K.E gain} \quad \Rightarrow \quad mgh = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{2gh}$$

Example 6.8 A marble is projected vertically upwards with an initial velocity u. The average resistive force acting is f. How do you determine the maximum height reached by the marble?

K.E. loss = G.P.E. gain + energy loss due to resistance

$$\frac{1}{2}mu^2 - 0 = mgH_{\max} + fH_{\max} \quad \Rightarrow \quad H_{\max} = \frac{mu^2}{2(mg+f)}$$



Example 6.9 A box of mass m slides down along a slope that is inclined at an angle θ to the horizontal. There is a constant friction f acting on the box. When the box has moved through a distance of Δs down the slope from A to B, write down an equation relating its velocities v_A and v_B by applying the law of conservation of energy.

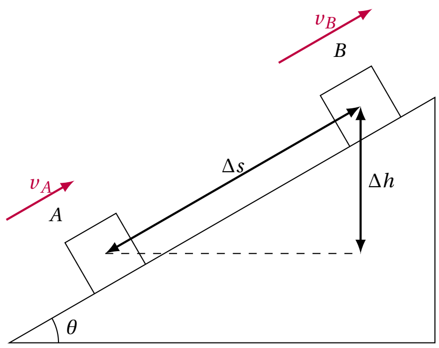
$$\begin{aligned}
\text{Total work done} &= \text{change in total energy}\\
-W_f &= \Delta E_k + \Delta E_p\\
-fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + \left(mgh_B - mgh_A\right)\\
-fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) - mg\Delta h\\
v_B^2 &= v_A^2 + 2g\Delta s\sin\theta - \frac{2fs}{m}
\end{aligned}$$

or equivalently, we can write

$$\begin{aligned}
\text{G.P.E loss} &= \text{K.E gain} + \text{energy loss due to friction}\\
mg\Delta h &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + fs\\
v_B^2 &= v_A^2 + 2g\Delta s\sin\theta - \frac{2fs}{m}
\end{aligned}$$

note that the two alternative ways of thinking produce the same result

Example 6.10 A slope is inclined at an angle θ to the horizontal. A box of mass m is pushed up the slope with a constant force F parallel to the slope, and the box experiences a constant frictional force f. When the box has moved through a distance of Δs along the slope from A to B, find an equation relating its velocities v_A and v_B.



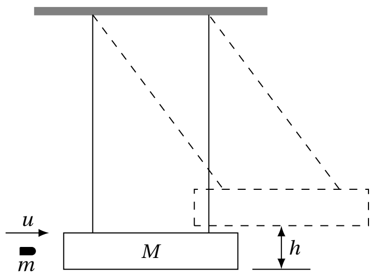
$$\begin{aligned}
\text{Total work} &= \text{change in total energy} \\
W_F - W_f &= \Delta E_k + \Delta E_p \\
Fs - fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + \left(mgh_B - mgh_A\right) \\
Fs - fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + mg\Delta h \\
v_B^2 &= v_A^2 + \frac{2(F-f)s}{m} - 2g\Delta s\sin\theta
\end{aligned}$$

or equivalently, we can write

$$\begin{aligned}
\text{work by } F &= \text{K.E gain } + \text{ G.P.E gain } + \text{ energy loss due to friction} \\
Fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + mg\Delta h + fs \\
v_B^2 &= v_A^2 + \frac{2(F-f)s}{m} - 2g\Delta s\sin\theta
\end{aligned}$$

again the two approaches produce the same expression for final velocity

Example 6.11 A *ballistic pendulum* is a device used to measure the speeds of fast-moving bullets. It consists of a large block of wood of mass M, suspended from two long light strings. A bullet of mass m is fired into the block, and the block and bullet combination swings upward. If the centre of mass rises a vertical distance h, what is the initial speed u of the bullet?



As th bullet enters block, the combined momentum is conserved:

$$mu = (M+m)v \quad \Rightarrow \quad v = \frac{mu}{M+m}$$

when the system swings upward, K.E. transforms into G.P.E. but total energy is conserved:

$$\frac{1}{2}(M+m)v^2 = (M+g)gh \quad \Rightarrow \quad v = \sqrt{2gh}$$

putting the two equations together, we find: $u = \left(1 + \frac{M}{m}\right)\sqrt{2gh}$

Power

To describe how fast work is done, we introduce the notion of power:

Power is defined as the work done per unit time:

$$P = \frac{\Delta W}{\Delta t}$$

The unit of poweris the Watt[7] $[P] = \frac{[W]}{[t]} = \text{J s}^{-1} = \text{W}$ (watt). In other words, if one joule of work is done in one second, the power is one **watt**
$P = \frac{\Delta W}{\Delta t}$ gives the *average* power during in a period of time Δt, to find the *instantaneous* power at a particular moment, we have

$$P = \frac{\Delta W}{\Delta t} = \frac{F\Delta s}{\Delta t} \quad \Rightarrow$$

$$P = Fv$$

[7] Which leads to the real *Joule* of a joke: Watt is the unit of poweer?...

Example 6.12 There are 150 steps to the top of a tower, and the average height of each step is 25 cm. It takes a man of 72 kg two minutes to run up all the steps. What is his average power?

$$P = \frac{\Delta E_p}{\Delta t} = \frac{mg\Delta h}{\Delta t} = \frac{72 \times 9.81 \times (150 \times 0.25)}{120} \quad \Rightarrow \quad P \approx 221 \text{ W}$$

Example 6.13 A turbine is used to generate electrical power from the wind. Given that the blades of the turbine sweep an area of 500 m^2, the density of air is 1.3 kg m^{-3}, and the wind speed is 10 m s^{-1}. Assume no energy loss, find the power available from the wind.

$$P = \frac{\Delta E_k}{\Delta t} = \frac{\frac{1}{2}\Delta m v^2}{\Delta t} = \frac{\frac{1}{2}\rho\Delta V v^2}{\Delta t} = \frac{\frac{1}{2}\rho A\Delta x v^2}{\Delta t} \quad \Rightarrow \quad P = \frac{1}{2}\rho A v^3$$

$$P = \frac{1}{2} \times 1.3 \times 500 \times 10^3 \approx 3.25 \times 10^5 \text{ W}$$

Example 6.14 A ship is cruising a a constant speed of 15 m s^{-1}. The total resistive force acting is 9000 N. What is the output power of this ship?

Constant speed so equilibrium between driving force and resistive force

$$P = Fv = fv = 9000 \times 15 \quad \Rightarrow \quad P = 1.35 \times 10^5 \text{ W}$$

Example 6.15 A car of mass 800kg accelerates from rest on a horizontal road. Suppose the engine provides a constant power of 24000 W, and the resistive force can be given by $f = 16v$, where v is the speed of the car in m s^{-1}. (a) What is the acceleration of the car when it is travelling at 15 m s^{-1}? (b) What happens to the car if it maintains this driving power?

The equation of motion for the car is: $F_{\text{net}} = F - f = ma \Rightarrow \frac{P}{v} - \alpha v = ma$
at $v = 15$ m s^{-1}: $\frac{24000}{15} - 16 \times 15 = 800a \Rightarrow a = 1.7$ m s^{-2}
as car's velocity v increases, driving force $F = \frac{P}{v}$ decreases, resistive force $f = \alpha v$ increases
so resultant force will decrease, the car will accelerate at a decreasing rate
eventually it reaches an equilibrium state where $F = f$, the car then travels at constant speed v_t

$$F = f \Rightarrow \frac{P}{v_t} = \alpha v_t \Rightarrow \frac{24000}{v_t} = 16v_t \Rightarrow v_t \approx 38.7 \text{ m s}^{-1}$$

Efficiency

efficiency of a system is given by: efficiency $= \frac{\text{useful energy output}}{\text{total energy input}}$, or

$$\eta = \frac{W_{\text{useful}}}{W_{\text{total}}}$$

since $\Delta W = P\Delta t$, efficiency can also be evaluated in terms of power:

$$\eta = \frac{P_{\text{useful}}}{P_{\text{total}}}$$

Example 6.16 A water pumping system uses 3.0 kW of electrical power to raise water from a well. The pump lifts 1500 kg of water per minute through a vertical height of 8.0 m. What is the efficiency of the sysem?

$$\eta = \frac{\Delta E_p}{\Delta E_{\text{in}}} = \frac{\Delta mgh}{P_{\text{in}}\Delta t} = \frac{1500 \times 9.81 \times 8.0}{3000 \times 60} \Rightarrow \eta \approx 65.4\%$$

Example 6.17 Water flows into a turbine from a reservoir at a vertical distance of 70 m above. The water flows through the turbine at a rate of 2500 kg per minute. What is the output power of the turbine if it is 85% efficient?

$$P_{\text{out}} = \eta P_{\text{in}} = \eta \frac{\Delta E_p}{\Delta t} = \eta \frac{\Delta mgh}{\Delta t} = 85\% \times \frac{2500 \times 9.81 \times 70}{60}$$

$$\Rightarrow P_{\text{out}} \approx 2.43 \times 10^4 \text{ W}$$

End-of-chapter questions

Work

Question 6.1 A trolley is pushed through a distance of 2.0 m with a force of F = 5.0 N along a track. The trolley experiences a constant frictional force of 3.0 N. What is the work done by F?

Question 6.2 A child of mass 40 kg slides down a slope from a height of 2.0 m above the ground. The slide is of a length of 6.0 m. How much work is done by gravity?

Question 6.3 A dog pulls on a lead with a force of 20 N at an angle of 20° to the horizontal. As the dog moves 10 m along the playground, find the work done (a) by the dog, (b) by the person holding the lead.

Question 6.4 A satellite is orbiting around the earth in a circular orbit due to gravitational attraction. The gravitational force on the satellite always acts towards the centre of the earth. Does this force do any work?

Question 6.5 A fixed mass of gas at a pressure of 1.50×10^5 Pa and initial volume of 2.80×10^{-4} m^3 is heated. The gas expands at a constant pressure to a final volume of 8.40×10^{-4} m^3. Find the work done by the gas.

Kinetic energy

Question 6.6 Estimate the kinetic energy of a family car travelling at 40 km per hour.

Question 6.7 Object B has double the mass and moves at twice the speed of object A. If the kinetic energy of object A is K, what is the kinetic enery of B?

gravitational potential energy

Question 6.8 Estimate how much gravitational potential energy you gain when you get to the top of the highest building in your country?

Question 6.9 Four uniform bricks, each of mass m and thickness H, are laid out on a table. In order to stack them on top of one another, how much work has to be done on the bricks?

work & energy transformations

Question 6.10 A stone is projected vertically upwards at a speed of 16 m s^{-1} from the ground. Air resistance is negligible. (a) What is the greatest height reached by the stone? (b) When the stone is at a height of 5.0 m, what is its speed? (c) If the stone is projected at an angle from the upward vertical with the same initial speed, what is its speed when it reaches a height ob 5.0 m?

Question 6.11 A hammer of mass 600 g hits a nail at a velocity of 10 m s^{-1}. The nail is pushed into a plank by 3.0 mm. What is the average frictional force acting on the nail?

Question 6.12 A block of mass m is pushed through a fixed distance along a horizontal frictionless surface by a constant force. Show that the final speed v of the block has: $v \propto \frac{1}{\sqrt{m}}$.

Question 6.13 A block of 3.0 kg is released from rest on a slope at an angle $\theta = \sin^{-1}\left(\frac{1}{5}\right)$ to the horizontal. As the block travels 6.0 m down the slope, it experiences a frictional force of 5.0 N. What is the final speed of the block?

Question 6.14 A cyclist is travelling up a hill at a constant speed. The cyclist uses 640 J of energy to travel a distance of 25 m. If the total resistive force opposing the motion is 9.0 N, what is the increase in gravitational potential energy? To determine the increase in the height, what further information do you need?

Question 6.15 A pendulum of mass 120 g is released from a position that is 1.6 cm above its equilibrium position. (a) At what point in its motion is the kinetic energy of a pendulum bob at its maximum? (b) At what point is the gravitational potential energy at a maximum? (c) When the kinetic energy is half its greatest value, how much gravitational potential energy does it have?

Question 6.16 A particle of mass m is initially at a height of h above the ground. It is then released from rest. Just before hitting the ground, the particle gains a speed of v. What is the average resistive force acting on the particle during the fall?

Question 6.17 Men's pole vault (a athletic event in which a person jumps over a bar with the aid of a long, flexible pole) world record is about 6 m. Could this record be raised to, say, 10 m by using a longer pole? If not, why is it impossible?

Question 6.18 This question concerns the design of a roller coaster. One designer says each summit must be lower than the previous one. Another designer suggests that it does not matter what heights the summits are as long as the first one is the highest. What do you think?

Power

Question 6.19 A car engine exerts an average force of 400 N in moving the car 900 m in 200 s. What is the average power developed?

Question 6.20 During a human heart beat, about 20 g of blood is pushed into the main arteries. This blood is accelerated from a speed of 0.20 m s^{-1} to 0.35 m s^{-1}. For a heart pulsing at 75 beats per minute, what is the average power developed by the heart?

Question 6.21 Estimate your body power when you run upstairs at full speed.

Question 6.22 A truck of mass 2700 kg is travelling at a constant speed of 9.0 m s^{-1} up a road that is inclined at 8.0° to the horizontal. Assume that the resistive forces are negligible. What is the useful power from the engine of the truck?

Question 6.23 Given that the forces resisting the motion of a racing car is proportional to the square of the car's speed. The car has an output power of 200 kW when travelling at a steady speed of 50 m s^{-1}. What output power is required to maintain a speed of 80 m s^{-1}?

Question 6.24 A car of mass 800 kg travels in a straight line up a slope. The total resistive force f_R can be modelled by the equation: $f_R = kv^2$, where constant $k = 5.0$ kg m^{-1} and v is the car's speed. When the car travels at a steady speed of 10 m s^{-1}, the engine exerts a force of 2.1 kN up the slope. (a) Find the component of the car's weight down the slope, and hence find the angle that the slope makes with the horizontal. (b) Find the power output from the engine. (c) If the car then travels onto a horizontal road. The engine's output power is unchanged and the resistive force obeys the same model as before. Find the acceleration of the car when its speed is 15 m s^{-1}. (d) The car eventually reaches a constant speed on the horizontal road. Find this terminal speed.

Efficiency

Question 6.25 An electric motor is used to lift a mass. When operating at full power, the current in the motor is 2.0 A and the voltage is 5.0 V. If the motor is 50% efficient, what is the time take to lift a mass of 400 g through a height of 2.5 m?

Question 6.26 A turbine at a hydroelectric power station is designed to drive a generator to produce electrical energy. The water falls through a vertical distance of 8.0 m at a rate of 150 kg per second as it passes through the turbine. The generator supplies a current of 24 A at a voltage of 220 V. What is the efficiency of the turbine system?

Question 6.27 When power to the grid is not required during the night, 200 MW of electrical power is used to pump water from a reservoir up to a lake 300 m higher. The pumping system operates at an efficiency of 90%. What mass of water can be pumped to the lake in three hours?

Question 6.28 Given that the petrol engine for a real car is 25% efficient. The fuel consumption for the engine is 16 litres per hour, and the energy density of the fuel is 44 MJ per litre. What is the power output from the engine?

Question 6.29 A bow shoots an arrow of mass 150 g vertically upwards with an initial speed of 30 m s^{-1}. The potential energy stored in the bow before release is 200 J. The arrow reaches a height of 30 m above the point of release. (a) What is the energy loss just after the arrow is released? (b) What is the efficiency of the bow for converting its potential energy into useful kinetic energy? (c) What is the energy loss due to air resistance? (d) What is the average force due to air resistance?

7 Materials

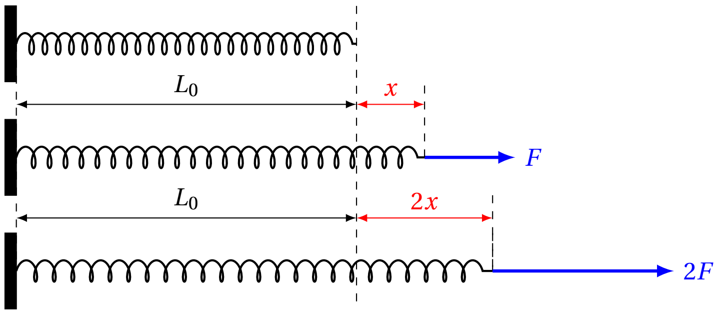
In this chapter, we study how materials behave under a range of conditions. An important notion is the elasticity of materials, for **elastic** materials, when external force is removed, the material returns to its original shape. If the material cannot restore to original shape, it is said to be **inelastic**, or **plastic**

Springs

Hooke's law

When a force F is applied to a spring, it is stretched from original length L_0 to some length L. This extension of the spring, $x = L - L_0$, is proportional to the force applied.

> Extension of an ideal spring is directly proportional to the load applied (within a certain range), this is called **Hooke's law**.



Hooke's law can be summarised by the equation:

$$F = kx$$

The proportionality constant k is called the **spring constant**, a larger k means a greater force is required to extend the spring by same amount.
A spring with a large k is said to be *stiff*. Note that we usually fall into the trap of thinking about materials under tension, however, Hooke's law also holds if spring is being compressed - i.e., if a spring is pushed to have compression of x, we still have force applied $F = kx$.
The linear relationship between F and x is only true up to the limit of the material capability. The point beyond which Hooke's law no longer

holds is called the **limit of proportionality**. Beyond this point it will still stretch, (and return to the origin if unloaded) but not in a linear fashion. If the force on the spring is greater still then, spring might be overstretched and no longer exhibits elastic behaviour - it it may become permanently deformed. The point beyond which spring cannot return to original length is called the **elastic limit** and the deformation called **permanent set.** [1]

[1] The elastic limit of a spring and the limit of proportionality are two different but always confused concepts. These two limits are usually very close to one another, but they are conceptually different.

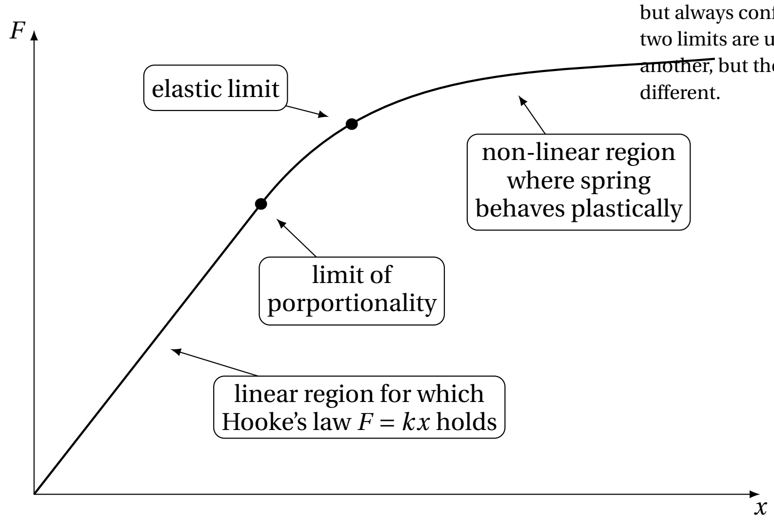


Figure 7.1: force-extension graph for a typical spring under load

Example 7.1 A spring has a natural length of 20.0 cm. When a mass of 250 g is suspended from the spring, the new length of the spring is 26.0 cm. Find the spring constant.

$$k = \frac{F}{x} = \frac{mg}{L - L_0} = \frac{0.250 \times 9.81}{(26.0 - 20.0) \times 10^{-2}} \quad \Rightarrow \quad k \approx 40.9 \text{ N m}^{-1}$$

Example 7.2 A spring has a spring constant of 270 N m^{-1}. A mass of 1.2 kg is hung from the spring. When the mass is released from a position where the spring has an extension of 5.0 cm, what is the acceleration of the mass?

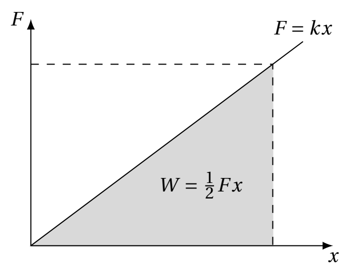
$$F_{net} = F - mg = kx - mg = ma \quad \Rightarrow$$

$$a = \frac{kx - mg}{m} = \frac{270 \times 5.0 \times 10^{-2} - 1.2 \times 9.81}{1.2} \approx 1.44 \text{ m s}^{-2}$$

Elastic potential energy in a spring

We stretch a spring by exherting a force over a distance. It follows then that to stretch or compress a spring, work must be done. The work done stored is called *elastic potential energy*.

Note that force in spring varies as spring is stretched so to find work in stretching a spring by x, we cannot just multiply it by F as one would do



with a constant force doing work. Instead we compute the area under F-x graph[2]. The graph is a right-angled triangle so:

$$W = \frac{1}{2}Fx = \frac{1}{2}kx^2$$

The formula for the elastic potential energy stored in a spring is:

$$E_p = \frac{1}{2}kx^2$$

[2] Mathematically, we can also integrate over the total extension to find this work done $W = \int_0^x F\,\mathrm{d}x = \int_0^x kx\,\mathrm{d}x = \frac{1}{2}kx^2$, which of course gives the same result.

Example 7.3 A steel spring has a spring constant of 20 N cm^{-1}. How much work is needed to stretch it from an extension of 3.0 cm to an extension of 5.0 cm?

Work done needed equals the increase in elastic potential energy:

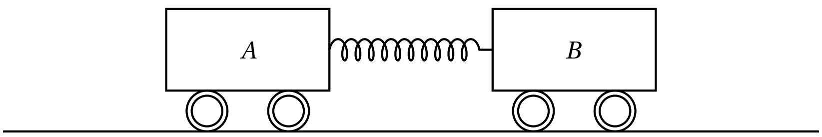
$$W = \Delta E_p = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 = \frac{1}{2} \times 2000 \times (0.050^2 - 0.030^2) = 1.6\ \mathrm{J}$$

Example 7.4 A trolley of 400 g can travel freely along a horizontal surface. It is pushed against a spring buffer. Suppose the spring is initially compressed by 5.0 cm under a 20 N force. When the trolley is released, it accelerates until it becomes detached. What is the trolley's final speed?

Elastic potential energy in spring $\Rightarrow$ kinetic energy of trolley

$$\frac{1}{2}Fx = \frac{1}{2}mv^2 \quad \Rightarrow \quad \frac{1}{2}\times 20\times 0.050 = \frac{1}{2}\times 0.40\times v^2 \quad \Rightarrow \quad v \approx 1.58\ \mathrm{m\,s^{-1}}$$

Example 7.5 The same spring is now set between two trolleys A and B of mass 400 g and 600 g. Initially the spring is again compressed by 5.0 cm under a force of 20 N. After both trolleys are released, what are their final speeds?



Elastic potential energy in spring transforms into kinetic energy of the two trolleys:

$$\frac{1}{2}Fx = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 \quad \Rightarrow \quad 20 \times 0.050 = 0.40 v_A^2 + 0.60 v_B^2$$

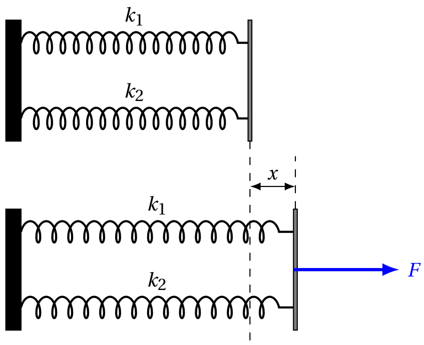
Total momentum for trolley A and B as a whole is conserved:

$$m_B v_B - m_A v_A = 0 \quad \Rightarrow \quad 0.40 v_A = 0.60 v_B$$

solving the simultaneous equations, we find: $v_A \approx 1.22\ \mathrm{m\,s^{-1}}$, $v_B \approx 0.82\ \mathrm{m\,s^{-1}}$

Spring combinations

So far we have discussed the properties of a single spring, next we investigate how a set of springs respond to a given load.



Parallel springs

Let's take two springs connected in parallel, when the combination is stretched under a load of F, extension in each spring should be the same:

$$x_1 = x_2 = x$$

the forces in each spring will in general be different, but sum of these must be equal to load:

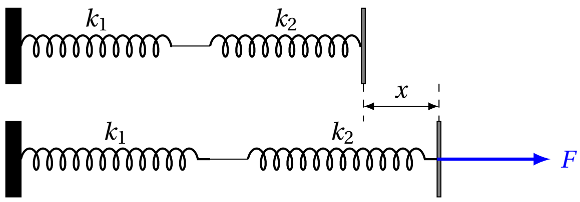
$$F = F_1 + F_2$$

Divide both sides by x, we have: $\frac{F}{x} = \frac{F_1}{x_1} + \frac{F_2}{x_2}$.
Recall the Hooke's law, this becomes: $k = k_1 + k_2$
Generalize for n springs in parallel connection, the combined spring constant is

$$k = k_1 + k_2 + \cdots + k_n$$

Series springs

Let's now take two springs in series



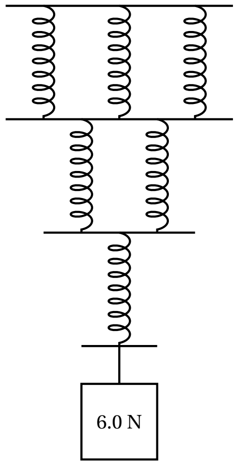
The force in each spring is the same: $F_1 = F_2 = F$
But total extension is the sum of individual extensions: $x = x_1 + x_2$.
Divide by the same F, we have: $\frac{x}{F} = \frac{x_1}{F_1} + \frac{x_2}{F_2}$.
For combined spring constant, we find: $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$
For n springs connected in series, the combined spring constant is therefore given by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots \frac{1}{k_n}$$

If we set up n springs in parallel, we actually make it thicker, it then requires a stronger force to stretch it by the same extension so we indeed see the combined spring constant is greater than that of any individuals. If we set up n springs in series, we make it longer, instead of pulling one spring at a time, the same force now stretches n springs simultaneously, this gives rise to a greater total extension so the combined spring constant must be less than that of any individual.
Example 7.6 (a) A spring with $k_1 = 20\ \mathrm{N\,cm^{-1}}$ is connected in series with a second spring with $k_2 = 30\ \mathrm{N\,cm^{-1}}$. When a force of 60 N is applied, what is the total extension of the combination? (b) The same two springs

are now connected in parallel. When a force of 50 N is applied on the combination, what is the extension?

For series connection: $k = \left(\frac{1}{k_1} + \frac{1}{k_1}\right)^{-1} = \left(\frac{1}{20} + \frac{1}{30}\right)^{-1} = 12\ \text{N cm}^{-1} \Rightarrow x = \frac{F}{k} = \frac{60}{12} = 5.0\ \text{cm}$
or sum up extension of each spring: $x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} = \frac{60}{20} + \frac{60}{30} = 5.0\ \text{cm}$
for parallel connection: $k = k_1 + k_2 = 20 + 30 = 50\ \text{N cm}^{-1} \Rightarrow x = \frac{F}{k} = \frac{50}{50} = 1.0\ \text{cm}$ □



Example 7.7 A set of identical springs are set up as shown. Each individual spring extends by 1.0 cm under a load of 1.0 N. Assume the limit of proportionality is not exceeded, what is the total extension for this combination when a load of 6.0 N is applied?

For a single spring: $k = 1.0\ \text{N cm}^{-1}$, the combined spring constant: $k_{\text{total}} = \left(\frac{1}{3k} + \frac{1}{2k} + \frac{1}{k}\right)^{-1}$

$$k_{\text{total}} = \left(\frac{1}{3.0} + \frac{1}{2.0} + \frac{1}{1.0}\right)^{-1} = \frac{6}{11} \approx 0.545\ \text{N cm}^{-1}$$

Total extension: $x = \frac{F}{k_{\text{total}}} = \frac{6.0}{0.545} = 11\ \text{cm}$
Alternatively, we can find and add extensions of each layer. In particular, the top layer withstands a force of 6.0 N shared by three springs, so each spring has a force of 2.0 N, extension is 2.0 cm. Similarly, the other two layers extend by 3.0 cm and 6.0 cm respectively, hence, total extension: $x = 2.0 + 3.0 + 6.0 = 11$ cm

Stress, strain & Young modulus

From daily experience, it is easier to stretch a longer wire than a shorter one, the same tensile force also produce greater effects on a thinner material than on a thicker one. To accurately compare materials, we need better quantities to describe what's going on. To study a material's response to a tensile force, several new quantities are to be introduced:

Stress & Strain

Tensile strain is defined as the ratio of the extension of a wire to its natural length:

$$\epsilon = \frac{x}{L}$$

Tensile stress is defined as the force applied per unit cross-sectional area:

$$\sigma = \frac{F}{A}$$

Strain is the ratio of two lengths, so it is unit free. It gives us a measure comparable to extension that allows us to compare materials of different length.
Stress is comprable to pressure - by considering the force per unit area we have a measure that allows us to compare materials of different (uniform) cross-section. The units of stress, like pressure: $[\sigma] = \mathrm{N\,m^{-2}} = \mathrm{Pa}$ (pascal)
The elastic behaviour of any material, though the physical mechanism might be different, can be accurately modelled as a spring. Hooke's law says $F \propto x$, it then follows that $\frac{F}{A} \propto \frac{x}{L}$, so $\sigma \propto \epsilon$
i.e., stress and strain should also be proportional to each other within certain limit
Example 7.8 A copper wire has a cross-sectional area of $1.5 \times 10^{-6}\ \mathrm{m^2}$. The breaking stress of the wire is 2.0×10^8 Pa. Find the breaking force.

$$\sigma = \frac{F}{A} \quad \Rightarrow \quad F = \sigma A = 2.0 \times 10^8 \times 1.5 \times 10^{-6} = 3.0 \times 10^2\ \mathrm{N}$$

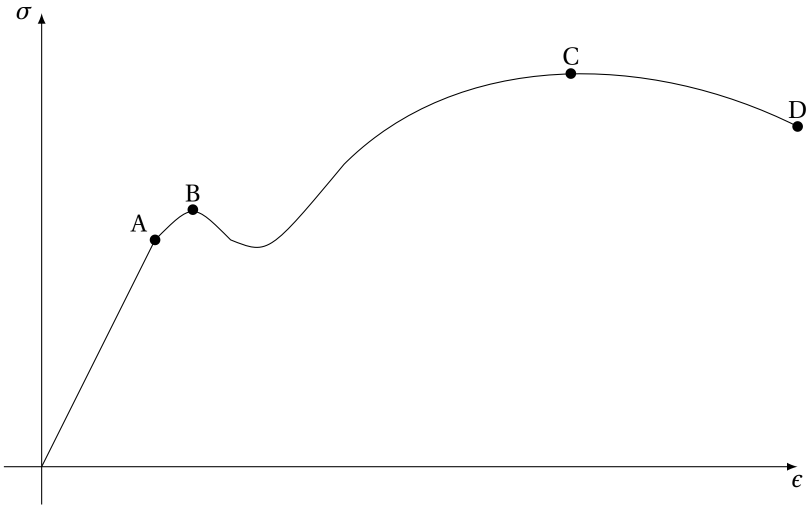
Young modulus

The ratio of stress to strain of a material is called the **Young modulus**:

$$E = \frac{\sigma}{\epsilon}$$

Because one of the parts of this is dimensionless, and the other pascals, the unit of measurementis: $[E] = \mathrm{Pa}$ (pascal)
Young modulus is a property of the material, for the same material, Young modulus is a constant, no matter in what shape it takes. i.e., it does not depend on the length or the cross section of the object.
Young modulus values can look "wrong" in answers to problems. This is because typical value of Young's modulus for metals: $E_{\mathrm{metal}} \sim 10^{11}$ Pa
Young modulus is a measure of the stiffness of a material - to produce same strain, greater stress is required for a material with greater Young modulus. You can get back to the spring constant from $E = \frac{\sigma}{\epsilon} = \frac{F/A}{x/L} = \frac{FL}{xA}$, we rearrange to get: $F = \frac{EA}{L}x$
Compare this with Hooke's law, we can identify the force constant to be given by: $k = \frac{EA}{L}$

– $E \nearrow \Rightarrow k \nearrow$, stiffer material makes stiffer springs
– $A \nearrow \Rightarrow k \nearrow$, more difficult to stretch a thick spring (think about parallel springs)
– $L \nearrow \Rightarrow k \searrow$, easier to stretch a long spring (think about series springs)



A The limit of proportionality
B The yield point
C The ultimate stress (maximium stress)
D The breaking point

Figure 7.2: Stress-strain curve for a ductile material

Figure 7.2 shows an example stress-strain curve. Note that the limit of proportionality is often a good approximation of the elastic limit of a metal. The "breaking stress" usually refers to the ultimate stress, i.e. the maximum stress the material can withstand, rather than the stress at the breaking point.

Example 7.9 A 200 N tensile force is applied on a steel wire of 1.5 m, the wire extends by 5.0 mm. The diameter of the cross section is 0.60 mm. What is the Young modulus of the steel wire?

$$E = \frac{F/A}{x/L} = \frac{FL}{Ax} = \frac{200 \times 1.5}{\pi \times (0.30 \times 10^{-3})^2 \times 5.0 \times 10^{-3}} \quad \Rightarrow \quad E \approx 2.1 \times 10^{11}\ \text{Pa}$$

Example 7.10 A copper wire of length 2.0 m is under a stress of 7.8×10^7 Pa. Given that the Young modulus of copper is 1.2×10^{11} Pa, what is (a) the strain of the wire, (b) the extension of the wire?

strain: $\epsilon = \frac{\sigma}{E} = \frac{7.8 \times 10^7}{1.2 \times 10^{11}} \quad \Rightarrow \quad \epsilon = 6.5 \times 10^{-4} = 0.065\%$

extension: $x = \epsilon L = 6.5 \times 10^{-4} \times 2.0 \quad \Rightarrow \quad x = 1.3 \times 10^{-3}$ m

Example 7.11 Several blocks of steel are used to support a bridge. Each block has a height of 30 cm and a cross section of 15 cm × 15 cm. The steel block is designed to compress 2.0 mm when the maximum load is applied. Given that the Young modulus of steel is 2.1×10^{11} Pa, what is the maximum load that can be supported by one block?

$$E = \frac{FL}{Ax} \quad \Rightarrow \quad F = \frac{EAx}{L} = \frac{2.1 \times 10^{11} \times (0.15 \times 0.15) \times 2.0 \times 10^{-3}}{0.30}$$

$$\Rightarrow \quad F \approx 3.15 \times 10^7\ \text{N}$$

Example 7.12 Two metal wires A and B are of the same length and they extend by the same amount under the same load. Given that Young modulus of wire A is twice of B, what is the ratio of their diameters?

$$E = \frac{FL}{Ax} \quad \Rightarrow \quad A = \frac{1}{4}\pi d^2 = \frac{FL}{Ex} \quad \Rightarrow$$

$$d^2 \propto \frac{1}{E} \quad \Rightarrow \quad \frac{d_A}{d_B} = \sqrt{\frac{E_B}{E_A}} = \frac{1}{\sqrt{2}}$$

Example 7.13 A full-size crane is ten times greater than a model crane in all linear dimensions. If they are made of the same materials, what is the ratio of the cable's extension?

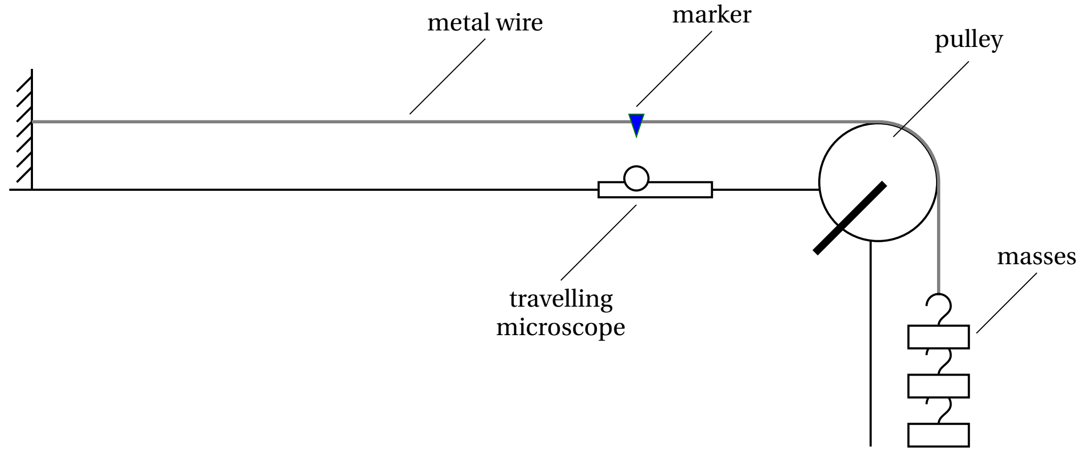
same material means same density ρ and same Young's modulus E, so:

$$E = \frac{FL}{Ax} \quad \Rightarrow \quad x = \frac{FL}{AE} = \frac{mgL}{\pi r^2 E} = \frac{\rho V g L}{\pi r^2 E} \quad \Rightarrow$$

$$x \propto \frac{VL}{r^2} \propto \frac{l^3 l}{l^2} \propto l^2 \quad \Rightarrow \quad \frac{x_{\text{full}}}{x_{\text{model}}} = 10^2 = 100$$

Measurement of Young modulus

The measurement of the Young's modulus of a metal wire is a required practical with a lot of detail. The experimental setup can be laid out as shown:



The original length L (up to the marker) of the wire is measured with *metre rule* ± 1mm. The diameter d of the wire is measured with *micrometer* ± 0.01mm, then cross-sectional area is: $A = \frac{1}{4}\pi d^2$.
For a recorded mass m attached to the wire, then force applied is $F = mg$
The extension x of the wire is taken to be distance moved out by the marker. In the lab we typically use a meter rule but in some textbooks you'll see this can be measured with a *travelling microscope*. A travelling microscope is basically a microscope that can move back and forth along a rail. The position of the microscope can be varied by turning a screw. This position can be read off a vernier scale. So in short, a travelling microscope can be used to measure the change in length with a very high resolution (typically to a precision of 0.01mm or 0.02 mm).
Once the data is collected, the stress can be calculated by $\sigma = \frac{F}{A}$, and strain can be calculated by $\epsilon = \frac{x}{L}$.
A graph of stress against strain can be plotted, a best fit line can be drawn and by taking the gradient of the straight-line section, the Young modulus can be obtained.

Stress-strain curves

Stress-strain curve for a material can be obtained using the methods in §7.13

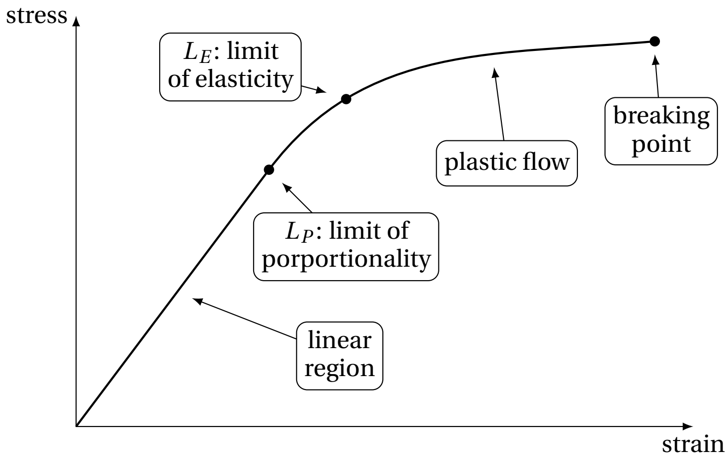


Figure 7.3: stress-strain graph for a typical metal

Ductile and plastic materials show some plastic deformation before breaking. Strictly speaking, ductility is continued plastic deformation at lower forces. Brittle materials on the other hand, break before any plastic deformation.

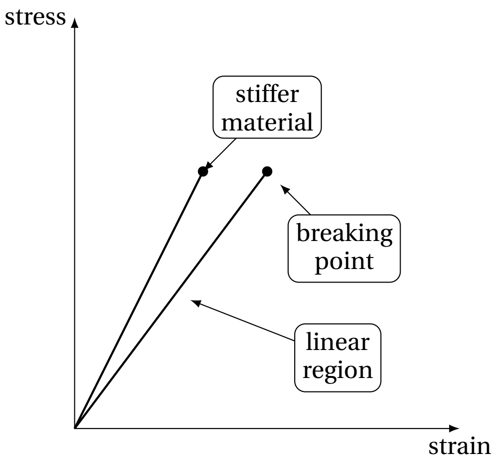
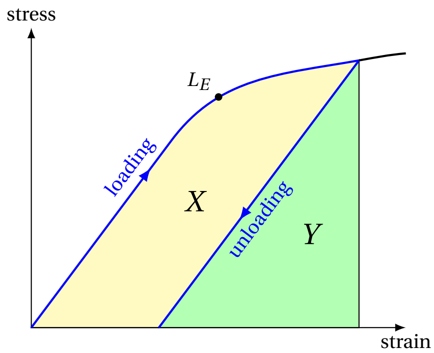


Figure 7.4: stress-strain graph for two brittle materials

For all materials, up to limit of proportionality L_E, stress is proportional to strain and the Young modulus can be calculated by the gradient of the line before L_P. In addition, consider the product of stress and strain: $\sigma \cdot \epsilon \sim \frac{F}{A}\frac{x}{L} \sim \frac{Fx}{AL} \sim \frac{W}{V}$. The area under stress-strain curve therefore, gives the work done per unit volume to stretch the wire. Up to limit of elasticity L_E, metal wire can return to original length when it is unloaded.
If a wire is stretched beyond the elastic limit, it follows a different path when force is removed and displays permanent set:
As discussed above, the area under stress-strain graph relates to energy, so the work done in stretching wire is given by $X+Y$. The energy returned when wire contracts is Y. The difference X gives energy loss as heat in the wire.



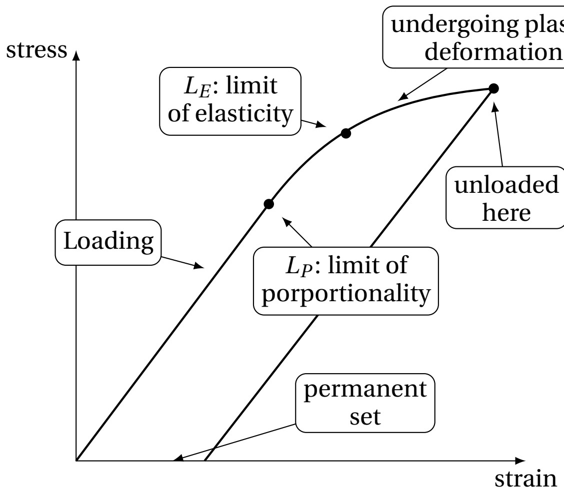


Figure 7.5: stress-strain graph for a material displaying permanent set

Metals

Metals consist of positive ions in a sea of delocalised electrons. During the elastic phase of deformation the spaces between the ions get larger and smaller. The metallic bonds resist this change from their equilibrium length and act like small springs acting to return the spacing to its original length.

An initial expectation of the plastic phase of deformation in a metallic lattice may be that the planes of ions slip past one another; however an analysis of the forces required for such movement gives an answer hundreds of times higher than the measured yield stress. Instead, the plastic deformation of metals must be explained in terms of *dislocations*. A dislocation occurs when there is a gap in the metallic lattice. Dislocations occur naturally in materials and enable plastic deformation to occur through the breaking of individual bonds in succession, rather than all at once.

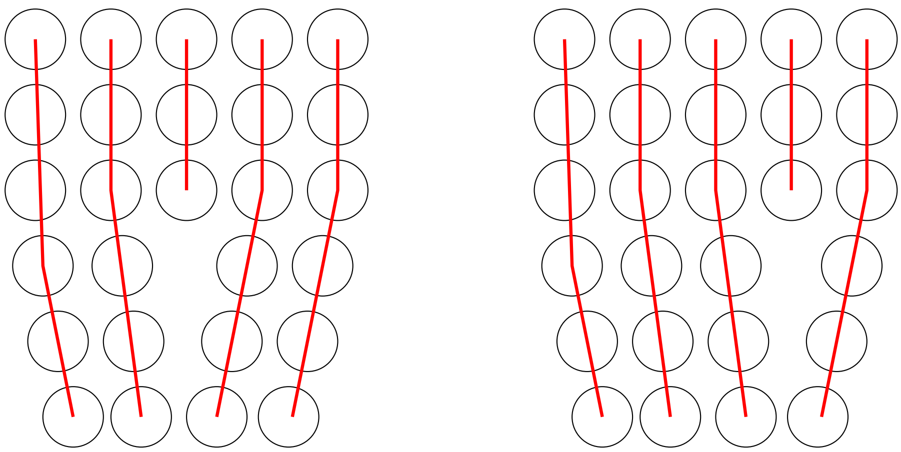


Figure 7.6: The movement of a dislocation

Figure 7.6 shows a dislocation moving within a metal which would allow the metal to deform by moving one atom at a time. As the movement of dislocations is the dominant mode of plastic deformation, changes to the ability of dislocations to move through the metal have significant effects on it properties. For example:

Work Hardening As a metal is deformed, the dislocations move through the structure. Slowly the dislocations reach grain-boundaries or other dislocations and are no longer able to move. The metal therefore becomes less ductile and more brittle. This may be a desired property in order to harden a metal, or the additional brittleness may be undesirable.

Alloying The addition of alloying atoms to the lattice can 'pin' a dislocation in place (as shown in figure 7.7. The metal is therefore no longer able to deform by the movement of dislocations so the metal has a greater yield stress and is less ductile. Examples include adding carbon to iron to produce steel or adding zinc to copper to produce brass.

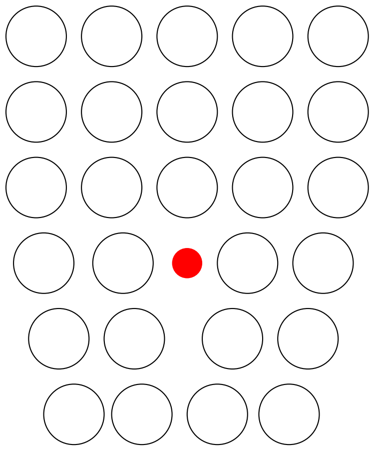


Figure 7.7: Alloying atom pinning a dislocation

The effect on a stress-strain graph can be seen in figure 7.8 below.

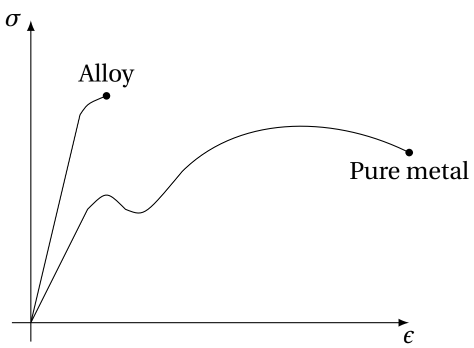


Figure 7.8: Stress-strain curve for a ductile material

Polymers

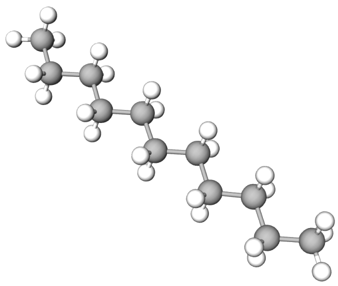


Figure 7.1: Long chain molecule of polyethylene, a common material.

Polymers consist of long chain molecules
weakly held together by inter-molecular forces. Initially the molecules are likely to be tangled-up together. As force is applied it is initially difficult to move the polymer chains from this state (**A**). As the chains begin

to unravel they straighten out by bond rotation, requiring relatively little force for a large increase in strain (**B**). As the polymer chains become straight it becomes much more difficult to extend the material any further without damaging the material (**C**).

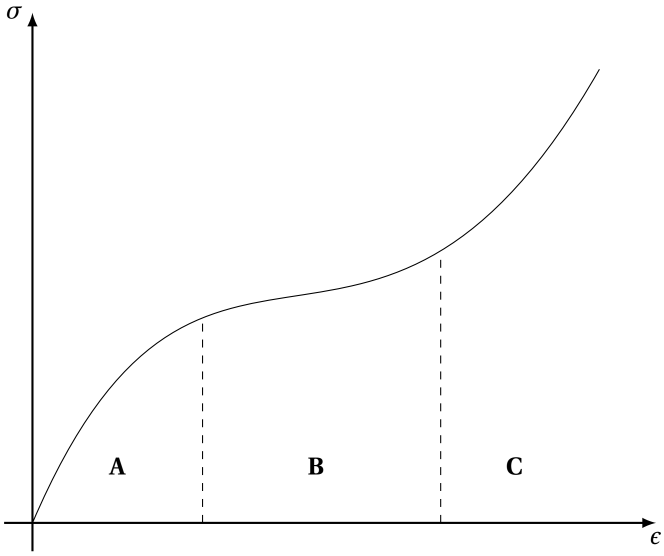


Figure 7.9: Stress-strain curve for a polymer

Rubber

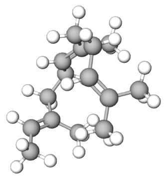
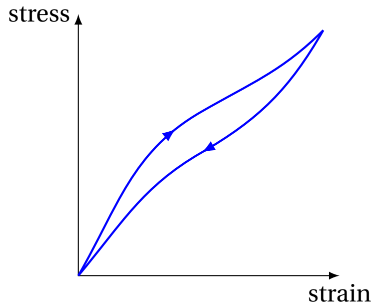


Figure 7.2: The twisted structure of natural rubber

When a rubber is exposed to stress or strain energy, internal rearrangements such as rotation and extension of the polymer chains occur. These changes occur as a function of the energy applied and the duration and rate of application, as well as the temperature at which the energy is applied. The difference between the two curves is the hysteresis losses.



End-of-chapter questions

Springs & Hooke's law

Question 7.1 A spring has a force constant of 500 N m^{-1}. The spring has a stretch length of 30 cm when a 40 N weight is hung from it. Find the natural length of the spring.

Question 7.2 Two springs, one with spring constant $k_1 = 8.0\ \text{N cm}^{-1}$ and the other with $k_2 = 12\ \text{N cm}^{-1}$. (a) When they are connected in parallel, what is the total extension when the combination supports a load of

60 N? (b) If they are connected in series, what is the total extension when they support the same load?

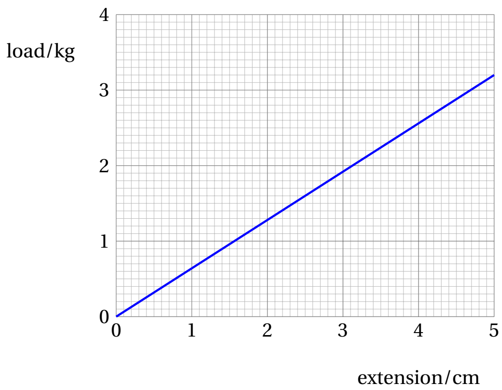
Question 7.3 A weight of 100 N is placed on top of a spring. The spring is compressed by 2.0 cm. Assume the spring obeys Hooke's law, how much strain energy is stored in the spring?

Question 7.4 A spring with a force constant 800 N m^{-1} is supported and stands vertically. A ball of mass 60 g falls vertically onto it. The ball has a speed of 3.0 m s^{-1} as it makes contact with the spring. Assume no energy loss, what is the maximum compression of the spring?

Question 7.5 A trolley of mass 160 g is placed on a frictionless track. The trolly pushed against a spring buffer of a force constant 90 N m^{-1}. The spring buffer is compressed by 7.0 cm. The trolley is then released from rest. (a) What is the initial acceleration of the trolley? (b) Assume no loss of energy, what is the final speed of the trolley along the track?

Question 7.6 Spring A is stiffer than spring B, i.e., $k_A > k_B$. On which spring is more elastic potential energy stored if they are stretched by (a) the same extension, (b) the same force?

Question 7.7 A load is attached to the lower end of a spring. The extension of the spring is measured when the load increases. The variation with extension of the load is shown. (a) Suggest whether the spring obeys Hooke's law. (b) Find the spring constant. (c) How much elastic potential energy is stored in the spring when the extension is 5.0 cm? (d) How much work is done to increase the extension from 2.0 cm to 4.0 cm?



Question 7.8 Suppose you have a spring, a ruler, a mass hanger and a set of masses. Suggest how the apparatus may be used to determine the load on the spring at (a) the limit of elasticity, (b) the limit of proportionality?

Stress, Strain, & Young modulus

Question 7.9 A cable of diameter 1.5 mm is under a tension of 200 N. Find the stress in this cable.

Question 7.10 Estimate the stress in your neck when it supports your head in a vertical position.

Question 7.11 A metal wire of natural length 1.8 m and diameter 0.70 mm is fixed to the ceiling at one end. When a mass of 6.5 kg is hung from the lower end, the wire extends by 2.7 mm. (a) Find the strain of the wire. (b) Find the Young modulus of the metal.

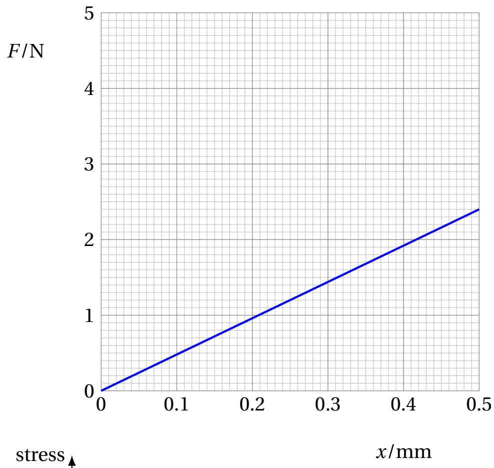
Question 7.12 A wire has a diameter of 0.50 mm with a Young modulus of 0.18 TPa. The length of the wire is increased by 0.20% by a force F. (a) Find the stress in the wire. (b) Find the force F.

Question 7.13 Two metal wires A and B are made of different materials. The diameter of wire A is twice that of wire B, and the Young modulus of wire A is three times that of wire B. If the wires are extended by the same strain, what is the ratio of the tension in wire A to tension in wire B?

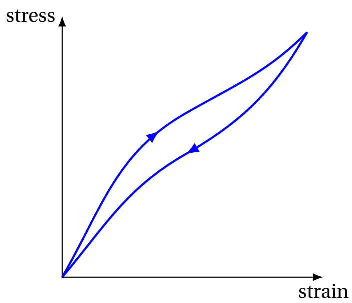
Question 7.14 Two rods A and B of the same diameter are joined end to end and hug vertically. Rod B is twice as long as rod A but has half the Young modulus. When a mass is hung from the combination, what is the ratio of the extension of rod A to the extension of rod B?

Question 7.15 Bones of different animals have very similar compositions. Suggest why heavier animals appear to have thicker bones?

Question 7.16 A load F is suspended from a copper wire. The graph shows the load-extension relation. (a) State two quantities other than F and x that are required to determine the Young modulus of copper. (b) Suggest how the two quantities may be measured. (c) Use the graph to find the energy stored in the wire when a load of 2.0 N is applied. (d) Given that steel has about twice the Young modulus as copper, sketch the variation with x of F for a steel wire that has the same dimensions as the copper wire.

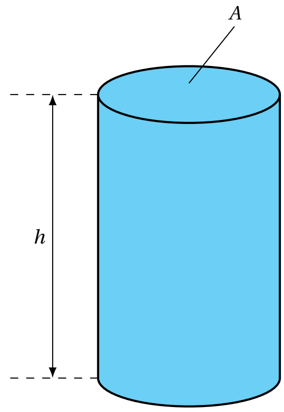


Question 7.17 In an experiment, a speciman of a rubber compound is being stretched and relaxed. The stress-strain curve is plotted. (a) State and explain whether this rubber compound behaves elastically. (b) The tyres on a vehicle are made of this rubber compound. Explain why the tyres become warm as the car travels on a road.



8 Fluids

a fluid, such as a liquid or a gas, is a substance that has no fixed shape
unlike a solid, a fluid can flow and yield easily under external force
in this chapter, we will study several aspects of a fluid



pressure in a fluid

at a depth of h below surface of a fluid, self-weight of the fluid could produce a pressure

$$p = \frac{F}{A} = \frac{mg}{A} = \frac{\rho V g}{A} \quad \Rightarrow \quad \boxed{p = \rho g h}$$

pressure in a liquid depends on depth
for different positions in a liquid, as long as they are at same depth, pressure is the same
i.e., pressure does not depend on volume or shape of container
atmospheric pressure also accounts for total pressure in a liquid
atmosphere presses on surface of a liquid, so total pressure at depth h is: $P = \rho g h + P_{\text{atm}}$
nevertheless, change in pressure still satisfies: $\Delta p = \rho g \Delta h$

Example 8.1 The atmospheric pressure is about 1.0×10^5 Pa. Given that the density of sea water is 1020 kg m^{-3}, what is the total pressure 50 m below the surface of the sea?

✔

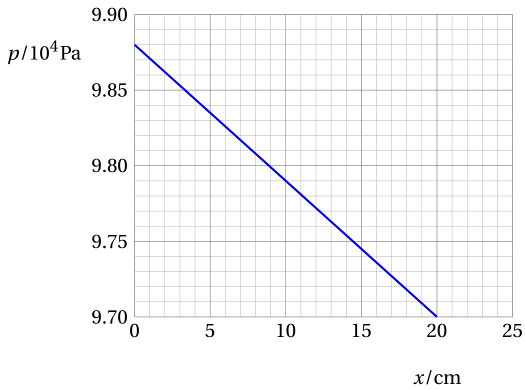
$$P = P_{\text{atm}} + \rho g h = 1.0 \times 10^5 + 1020 \times 9.81 \times 50 \quad \Rightarrow \quad P \approx 6.0 \times 10^5 \text{ Pa}$$

Example 8.2 A vertical column of liquid of height 10 m contains both oil and water. The pressure due to the liquids at the bottom of the column is 89 kPa. Given that the density of water is 1000 kg m^{-3} and the density of the oil is 840 kg m^{-3}. What is the depth of the oil?

✔

$$P = P_{\text{oil}} + P_{\text{water}} = \rho_o g h_o + \rho_w g h_w$$

$$840 \times 9.81 \times x + 1000 \times 9.81 \times (10 - x) = 89 \times 10^3 \quad \Rightarrow \quad x = 5.8 \text{ m}$$



Example 8.3 The pressure p of a liquid in a container varies with the height x above the base of the container as shown. The total depth of the liquid is 20 cm. (a) What is the atmospheric pressure? (b) What is the density of the liquid?

✔surface of liquid at height $x = 20$ cm, so

$$P_{\text{atm}} = 9.70 \times 10^4 \text{ Pa}$$

density of liquid: $\rho = \frac{\Delta p}{g \Delta h} = \frac{(9.88-9.70)\times 10^4}{9.81 \times 0.20} \quad \Rightarrow \quad \rho \approx 917 \text{ kg m}^{-3}$ □

pressure meters

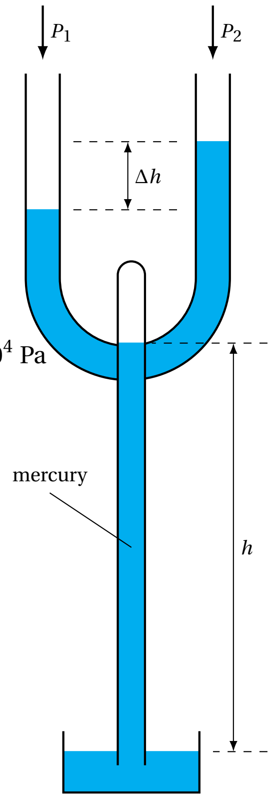
there are many types of instruments for pressure measurement
we would only focus on two: simple manometers and barometers [1]
they both use the fact that $\Delta p = \rho g \Delta h$ within a liquid

[1] Some examples of many other types of pressure gauges include
- mechanical gauges based on metallic pressure-sensing elements
- electronic gauges based on piezo-resistive effect
- hot-filament ionization gauges based on ion currents from a gas

Those who are interested are welcome to research into their functions and principles.

manometers

a **manometer** consists of a U-shaped tube filled with some liquid
any pressure difference between the two ends of the tube could cause a height difference between liquid levels
for the situation shown, at equilibrium, one has: $P_1 - P_2 = \rho g \Delta h$
if P_2 is a reference pressure, then P_1 can be calculated
though any fluid can be used in a manometer, *mercury* is preferred because of its high density ($\rho_{\text{Hg}} = 1.36 \times 10^4$ kg m^{-3})



Example 8.4 A manometer is used to measure the pressure of a gas supply. Side A of the tube is connected to the gas pipe, and the other side B of the tube is open to the atmosphere. If the mercury on side A is higher than on side B by 14 cm, what is the pressure of the gas? (density of mercury: 1.36×10^4 kg m^{-3}; atmospheric pressure: 1.01×10^5 Pa)

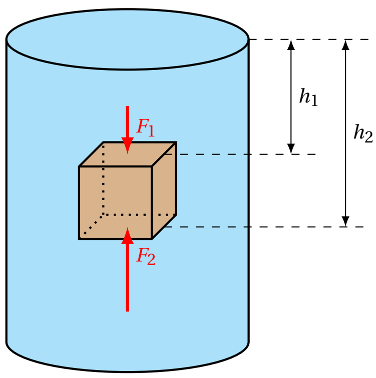
✔ $P_{\text{atm}} - P_{\text{gas}} = \rho g h \quad \Rightarrow \quad P_{\text{gas}} = 1.01\times10^5 - 1.36\times10^4 \times 9.81 \times 0.14 \quad \Rightarrow \quad P_{\text{gas}} \approx 8.23\times10^4$ Pa

barometers

take a long glass tube and fill it with mercury
let it stand upside down in a basin
there is atmospheric pressure pushing down on surface of mercury, so a height of mercury is supported up the tube
we can then compute atmospheric pressure by: $P_{\text{atm}} = \rho g h$
this instrument makes a **mercury barometer**

Example 8.5 If a mercury barometer supports a height of 760 mm of mercury above the fluid level in the container, what is the atmospheric pressure? If water is used as the barometric liquid, what is the minimum length of the tube required for the same atmospheric pressure? (density of mercury: 1.36×10^4 kg m^{-3}; density of water: 1.00×10^3 kg m^{-3};)

✔atmospheric pressure: $P_{\text{atm}} = \rho g h = 1.36 \times 10^4 \times 9.81 \times 0.760 \approx 1.01 \times 10^5$ Pa
if mercury is replaced by water: $h' = \frac{P_{\text{atm}}}{\rho' g} = \frac{1.01\times10^5}{1000\times9.81} \approx 10.3$ m □



upthrust

now consider a rectangular block immersed in a fluid
top and bottom surface are at different depths, so they experience different pressures
this gives rise to an overall upward force on the cylinder
this force is called the **upthrust**:

$$F_U = F_2 - F_1 = \rho g (h_2 - h_1) \times A \quad \Rightarrow \quad \boxed{F_U = \rho g V}$$

therefore upthrust exerted on an immersed object equals the weight of the fluid displaced

this is known as the **Archimedes' principle**
origin of upthrust: pressure difference between top and bottom surfaces
for an object of density ρ_o immersed in a liquid of density ρ_l
take force in downward direction to be positive, then resultant force acting is:

$$F_{net} = W - F_U = \rho_o g V - \rho_l g V = (\rho_o - \rho_l) g V$$

– if $\rho_o > \rho_l$, then $F_{net} > 0$, resultant force acts downwards, object will sink
– if $\rho_o < \rho_l$, then $F_{net} < 0$, resultant force acts upwards, object will rise
– if $\rho_o = \rho_l$, then $F_{net} = 0$, object is in equilibrium, it can float at that level

Example 8.6 A block of mass 80 g and volume 50 cm^3 is suspended from a string into water. When the block is fully immersed and kept at rest, what is the tension in the string?

✔
$T + F_U = W \quad \Rightarrow \quad T = mg - \rho g V = 0.080\times 9.81 - 1000\times 9.81\times 50\times 10^{-6} \quad \Rightarrow \quad T \approx 0.29$ N

end-of-chapter questions

Data for the questions below where applicable:
– density of water: 1.00×10^3 kg m^{-3}
– density of mercury: 13.6×10^3 kg m^{-3}
– atmospheric pressure: 1.0×10^5 Pa

pressure in a fluid

Question 8.1 (a) A dam holds a depth of 50 m of water. What is the water pressure at the base of the dam? (b) Suggest why the walls of a dam must be made thicker near the bottom?

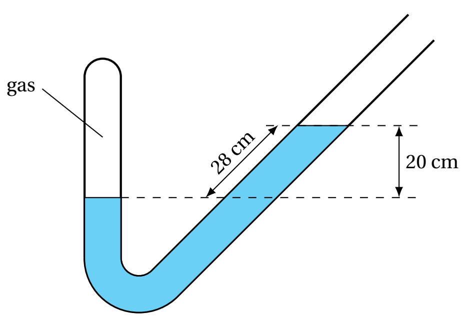
Question 8.2 The deepest trench on Earth is The Mariana Trench located in the Pacific Ocean. The maximum known depth is about 11 km. (a) Assume the sea water has a uniform density of about 1030 kg m^{-3}, estimate the pressure at the base of the trench. (b) The density of sea water actually increases slightly with pressure, suggest how this affects the result you have found.

Question 8.3 Instead of a large viewing window, the window of a submarine is usually of only a few centimetres in diameter. Why is the window made so small?

Question 8.4 If you punch several holes in the bottom of a container filled with water, water will spurt out due to the pressure. Now drop the container, suggest what will happen as it falls freely and defend your explanation.

Question 8.5 Air trapped inside a cylinder is attached to a U-shaped manometer containing mercury. The other side of the manometer is open to atmosphere. The mercury column on the side open to atmosphere is found to be 45 mm higher, what is the pressure of the trapped air?

Question 8.6 The figure shows a pipe closed at one end and open at the other end. Some gas is trapped by a column of mercury as shown. Find the pressure of the gas.



Question 8.7 A water manometer is used to measure the pressure created in a flexible container. Initially, the water columns on each side of the manometer is at the same level. When a girl stands on a 30 cm by 30 cm platform placed on top of the container, a height difference of 50 cm is observed in the manometer. (a) Find the pressure created by the girl. (b) Find the mass of the girl.

upthrust

Question 8.8 A lead block and an aluminium block of identical size are immersed in water. Upon which block is the upthrust greater?
Question 8.9 If somehow the gravitational field on the earth is increased, does a fish sink, float to the surface, or otherwise?
Question 8.10 A diver holds a cube of side 0.30 m and density 800kg m^{-3} near the seabed. (a) What is the upthrust on the cube? (b) What is its initial acceleration when the cube is released from rest?
Question 8.11 A solid sphere of radius 18 cm and density 2.4g cm^{-3} is fully submerged in water. A string pulls on the sphere so that the sphere does not sink. (a) Find the tension in the string. (b) If the string is cut, what is the instantaneous acceleration. (c) Describe and explain the acceleration of the sphere as it sinks.

9 *Waves*

A wave is a way of transferring and storing *energy* without the transport of matter. Most familiar examples are surface waves on water, sound waves, light waves. In the next two chapters, we are going to look at the basics of wave motion and some of the most important wave phenomena.
Wave motion is the propagation of disturbance – deviations from a state of equilibrium – from one place to another.

Wave terminology

Wave motion is the propagation of disturbance from one place to another

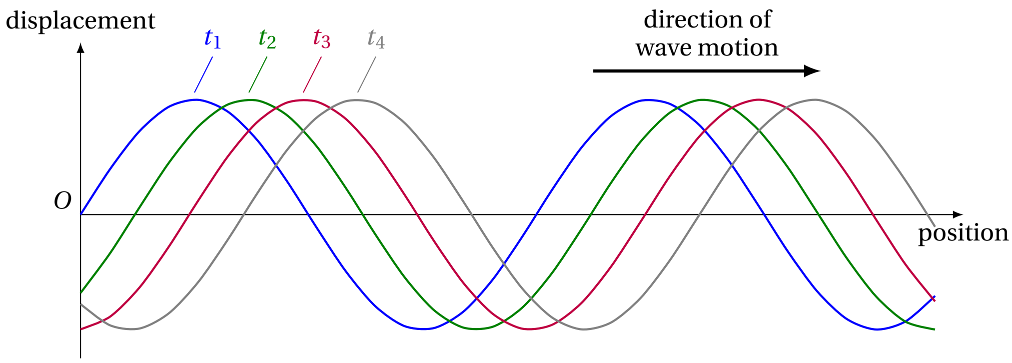


Figure 9.1: wave pattern at different times ($t_1 < t_2 < t_3 < t_4$) as wave travels in space

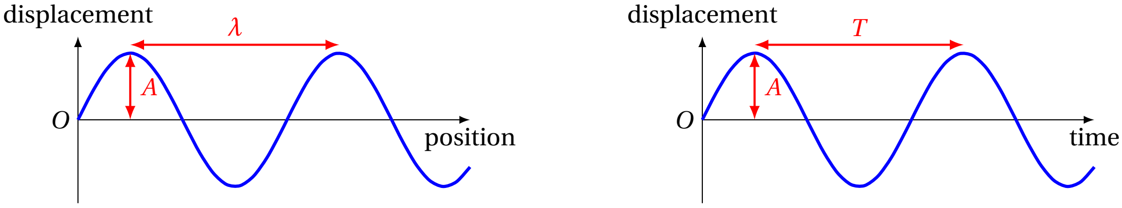
To describe the wave motion and particle vibrations, we can define the following quantities:
– as a wave moves from the source, each point oscillates back and forth about their rest positions. The distance from a particle's equilibrium position is called **displacement** of the particle . The greatest displacement for a particle is called the **amplitude** (A) .
A wave pattern repeats itself over a certain distance, the distance between two adjacent points undergoing exactly same motion is the **wavelength** (λ) . One can think of wavelength as crest-to-crest distance, trough-to-trough distance, etc.[1]
You can also think about how each point also repeats its vibrational motion over a certain time interval. The time for a particle to complete a full oscillation cycle is the **period** (T) . The number of oscillations for a particle per unit time is the **frequency** (f) which can also be defined as the number of crests passing a given point per unit time.

[1] For now, we take for granted that a wave is transverse. There are also longitudinal waves for which terms like crest and trough do not apply. We will get into that in §9.2.

The frequency of a wave is related to its period by

$$f = \frac{1}{T}$$

The unit of frequency: $[f]$ = Hz (hertz), where 1 Hz = 1 s^{-1}. There's some useful coded language here, many things are "per second", the "Hertz" part tell you it's probably a wave you're talking about.



wave pattern of all particles at one instant   vibration of one specific particle at all times

Wave energy is transferred along the direction of wave motion at a certain speed v. In one period, the wave moves forward by a distance of one wavelength, so **wave speed**

$$v = \frac{\lambda}{T}$$

, or in terms of frequency,

$$v = \lambda f$$

Example 9.1 When a wave travels on a water surface, the maximum depth of water is 21 cm and the minimum depth is 18 cm. What is the amplitude of the wave?

The amplitude is half the end-to-end distance: $A = \frac{1}{2}(21 - 18) = 1.5$ cm

Example 9.2 A wave travelling at 4.0 m s^{-1} has a wavelength of 50 cm, what is its period?

$$v = \frac{\lambda}{T} \quad \Rightarrow \quad T = \frac{\lambda}{v} = \frac{0.50}{4.0} = 0.125 \text{ s}$$

Transverse & longitudinal waves

A wave can either be *transverse* or *longitudinal*, depending on the direction of its oscillation

Transverse waves

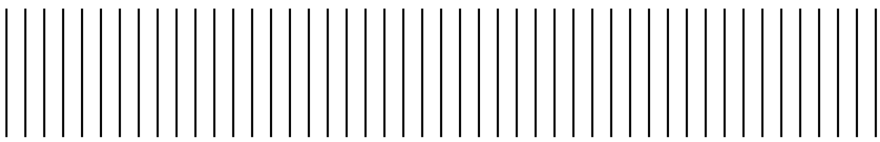
A **transverse** wave has vibrations at right angle to its direction of energy transfer .

Examples of transverse waves: – wave on a string, – surface wave on water, – light wave, etc.
For a transverse wave, greatest displacement in positive direction is called a *crest*, or a *peak*, the greatest displacement in negative direction is called a *trough*.

Longitudinal waves

A **longitudinal** wave has vibrations in parallel direction to energy transfer .

Examples of longitudinal waves: – sound waves, – wave along a stretched slinky, etc.
For a longitudinal wave, if medium gets squeezed, we say this region is a *compression*, if a medium expands, we say this region is a *rarefaction*.
The wavelength of a longitudinal wave can be defined as compression-to-compression distance.



All particles at their equilibrium position before a pressure wave is set up

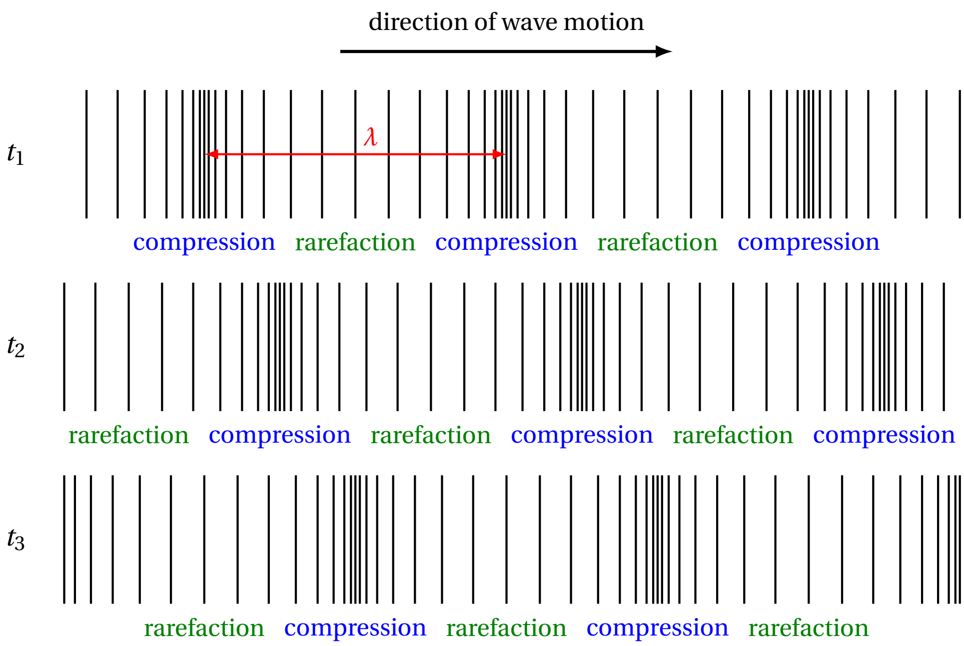


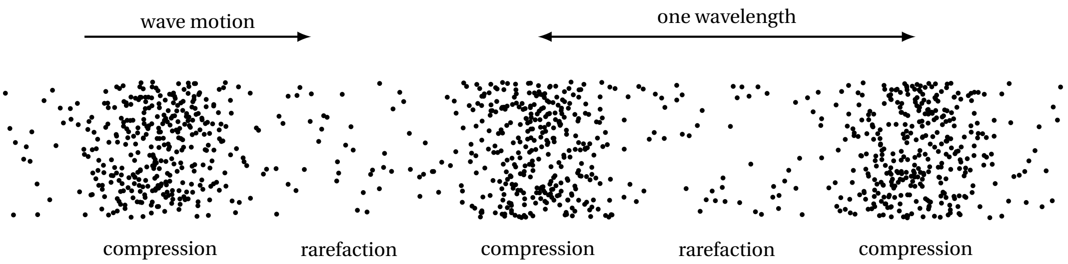
Figure 9.2: compression and rarefaction regions at different times ($t_1 < t_2 < t_3$) as a longitudinal pressure wave travels in space

Example 9.3 What is the distance between a compression and a rarefaction for a sound wave of frequency 550 Hz? (Speed of sound in air is about 330 m s^{-1}.)

$$d = \frac{1}{2}\lambda = \frac{v}{2f} = \frac{330}{2 \times 550} \quad \Rightarrow \quad d = 0.30 \text{ m}$$

Sound waves

Sound waves propagate via the compression and rarefaction of air (or other medium)
Molecules near vibrating source is pushed away from rest positions and into their neighbours, which then in turn push into their neighbours, and so on. In this manner the disturbance is transferred through the medium, forming a sound wave. The sound waves are longitudinal.



Propagation of sound waves require medium (air, water, steel, etc.) which means that sound cannot travel in vacuum. The speed of sound is material-dependent, but not frequency-dependent, for example, sound in general travels faster in denser medium.
– $v_{\text{air}} \approx 340$ m s^{-1} (under standard atmospheric pressure and room temperature)
– $v_{\text{water}} \approx 1500$ m s^{-1}
– $v_{\text{steel}} \approx 5000$ m s^{-1}
The *pitch* of a note is related to frequency of sound wave, rapid vibrations of sound source at high frequencies produce a high pitch. The *loudness* of sound mainly depends on amplitude of vibration. A greater amplitude means the wave is more energetic so it sounds louder.

Measurement of sound waves

Two key apparatuses for sound measurement are the **microphone** and the **oscilloscope**. Sound waves can be captured by a *microphone*, which converts sound into electrical signals, electrical signals can then be sent into an *oscilloscope* (see figure[2]) for measurement. The oscilloscope can be thought as an upgraded voltmeter that shows how voltage varies with time.
Oscilloscopes displays the variation of voltage (y-axis) with time (x-axis). The horizontal scale (time axis) is given by *time-base* settings, the vertical scale (voltage axis) is given by *voltage gain*, or *Y-sensitivity* settings. The

[2] The beautiful figure of the oscilloscope was created by *Hugues Vermeiren*, who generously shared the source codes on TeXample: `http://www.texample.net/tikz/examples/textronics-oscilloscope/`

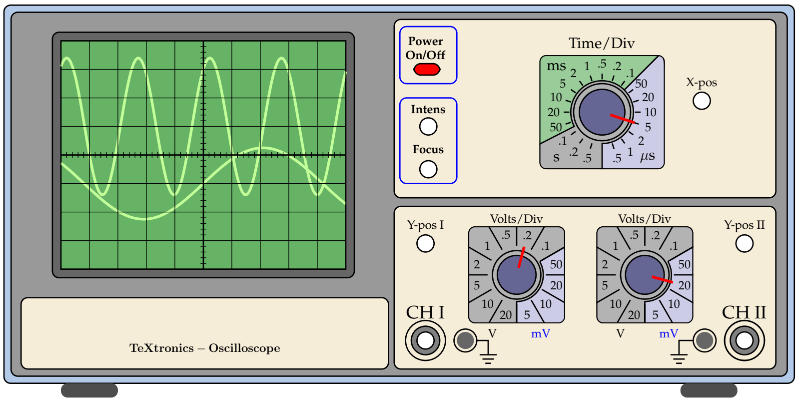
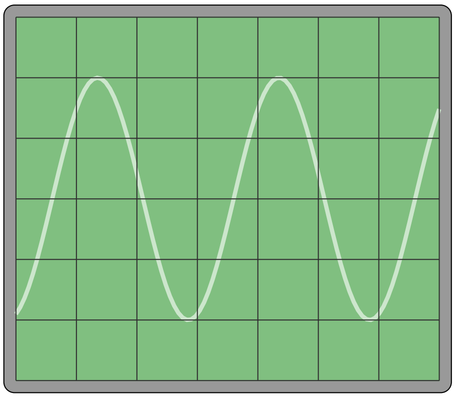


Figure 9.3: the display and controls of a typical cathode-ray oscilloscope

period T of the sound wave can be found using time-base settings and the frequency calculated by: $f = \frac{1}{T}$. The voltage amplitude can be found using the voltage gain.



Example 9.4 A sound wave is detected by a microphone and the trace is displayed on an oscilloscope as shown. If the time-base is set at 0.5 ms div^{-1} and the voltage gain is set at 2 V div^{-1}. What is the frequency and the amplitude of the signal?

period: $T = 3 \times 0.5\text{ ms} = 1.5 \times 10^{-3}\text{ s}$

frequency: $f = \frac{1}{T} = \frac{1}{1.5\times10^{-3}\text{ s}} \approx 667\text{ Hz}$

amplitude: $A = 2 \times 2\text{ V} = 4.0\text{ V}$

Electromagnetic waves

A wave can be either *mechanical* or *electromagnetic*, depending on whether it requires a medium. Waves that require a *medium* to travel are called **mechanical waves**, examples are sound waves, water waves, wave on a string, etc.
Mechanical waves on the other hand, involve vibration of matter particles. No medium is needed for propagation of **electromagnetic waves** i.e., they can travel in vacuum.

Properties of electromagnetic waves

– electromagnetic waves can travel in free space
– electromagnetic waves involve vibrations of electric and magnetic fields – an altering electric field can generate an altering magnetic field, which then further produces a new altering electric field, so on and so forth. Electric and magnetic fields then permeate through space, transferring energy and information.
– electromagnetic waves are all transverse.
– vibration of electric fields and magnetic fields are both perpendicular to wave motion.

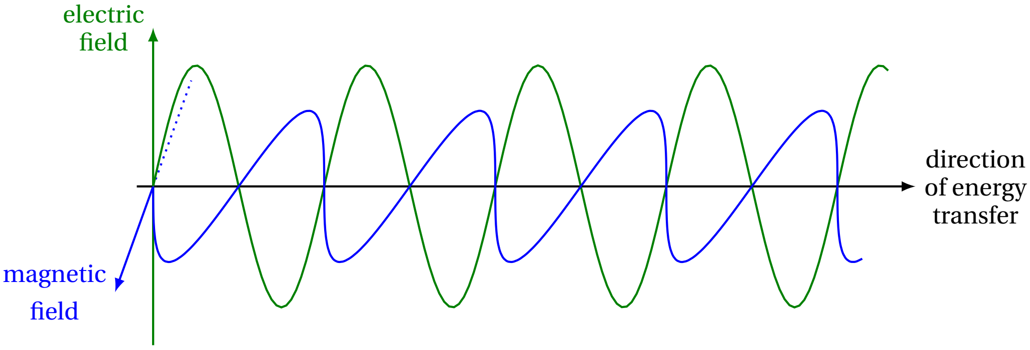


Figure 9.4: variation in electric and magnetic fields for an electromagnetic wave

Note that all electromagnetic waves travel at a constant speed $c = 3.0 \times 10^8$ m s^{-1} in free space i.e., speed of light in vacuum is constant[3]

[3] The speed of light in vacuum is actually a *universal* physical constant. According to Einstein's special relativity, this is the upper limit for the speed at which matter and information can travel. This speed is also independent of the inertial reference frame one chooses, i.e., the speed of light in vacuum is the same for all observers, regardless of the motion of the source or the observer.

The electromagnetic spectrum

Electromagnetic waves come in a wide range of wavelengths and frequencies, the distribution of electromagnetic radiation according to wavelength or frequency is the **electromagnetic spectrum** (see diagram)

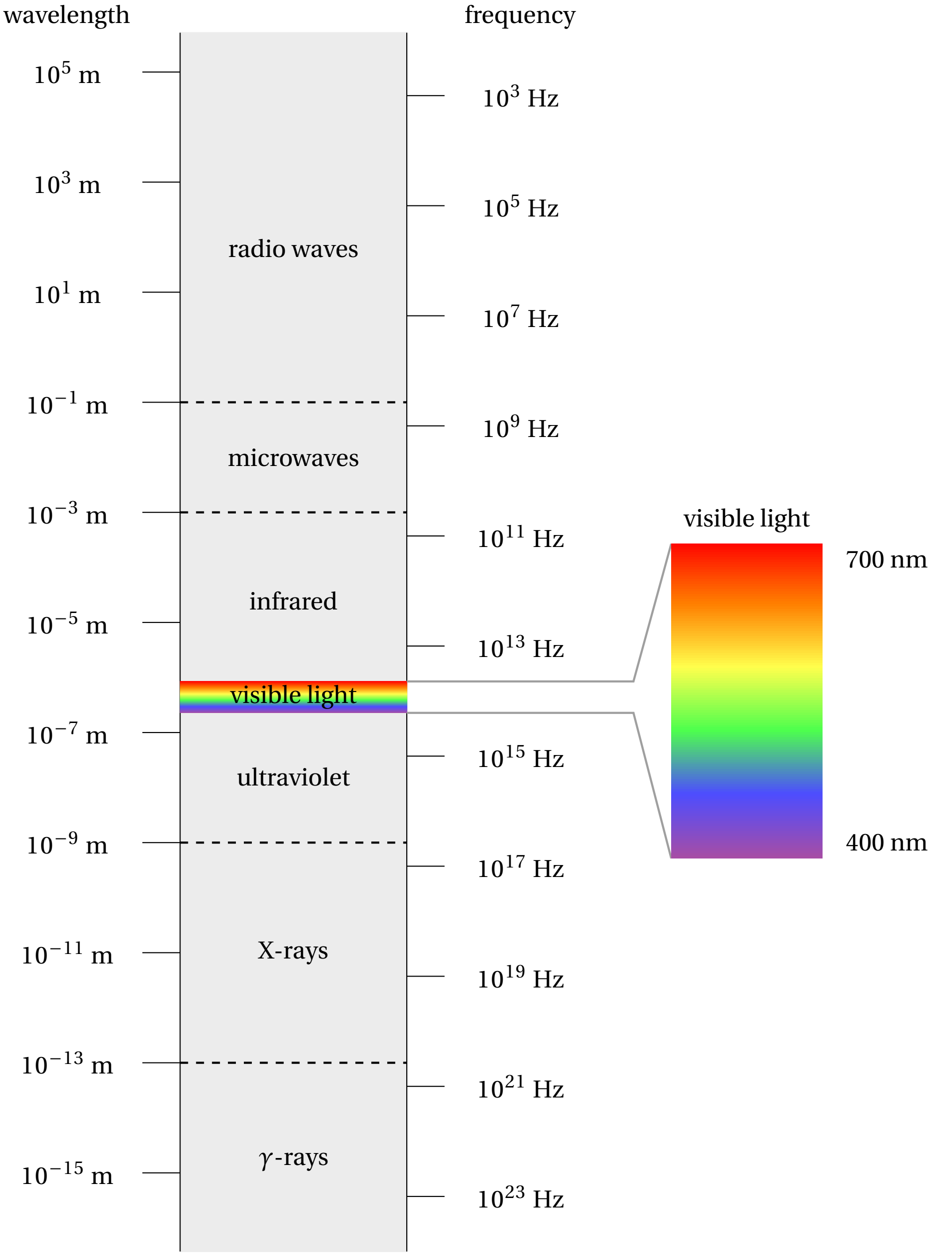


Figure 9.5: the electromagnetic spectrum

The table below shows electromagnetic spectrum in somewhat more precise details [4]

[4] There are no precise accepted boundaries between different ranges in the electromagnetic spectrum. The boundaries are actually somewhat ambiguous. The ranges of different portions tend to overlap. For example, a large portion of the ranges of X-rays overlap with that of γ-rays, and microwaves are considered by many people as a subdivision of radio waves. Therefore, the values given here are merely supposed to give you some rough idea about the order of magnitudes for electromagnetic wavelengths and frequencies. So the point I want to make here is: do not take the borderlines too seriously.

Region of spectrum	Range of wavelength	Range of frequency
radio waves	$10^{-1} \sim 10^{6}$ m	$10^{2} \sim 10^{9}$ Hz
microwaves	$10^{-3} \sim 10^{-1}$ m	$10^{9} \sim 10^{11}$ Hz
infra-red	$7\times10^{-7} \sim 10^{-3}$ m	$10^{11} \sim 10^{14}$ Hz
visible light	$4\times10^{-7} \sim 7\times10^{-7}$ m	$10^{14} \sim 10^{15}$ Hz
ultraviolet	$10^{-9} \sim 4\times10^{-7}$ m	$10^{15} \sim 10^{17}$ Hz
X-rays	$10^{-13} \sim 10^{-9}$ m	$10^{17} \sim 10^{19}$ Hz
γ-rays	$10^{-16} \sim 10^{-11}$ m	$10^{19} \sim 10^{24}$ Hz

Each type of electromagnetic waves has important applications in some area [5]

[5] The entries listed here only include a teeny-weeny part of the uses of electromagnetic radiation, somewhat based on my personal taste. I also included a handful of explanations for the examples that I found interesting (otherwise I would not choose them), as you will see a huge load of footnotes in the next few pages. You are encouraged to do some researches as well, I can guarantee that you will not be disappointed.

- Radio waves.
  Telecommunication (TV/radio broadcast, satellite communication). Having the longest wavelengths of all radiation, radio waves have the best ability to diffract around obstacles in city buildings and mountains, therefore a large area can be covered.
- Microwaves.
  Telecommunication (mobile phones, WiFi, Bluetooth, satellite communication). Microwaves do not diffract sufficiently as radio waves, but they can transmit more information per unit time because they have higher frequencies. Microwaves are used in short-range telecommunication, including mobile phones, wireless networks, and bluetooth connections.
  Heating food (microwave ovens). Frequency of microwaves are close to the natural frequencies of the rotational motion of water molecules. When food is exposed to microwaves, the water molecules in the food resonate and vibrate more violently, causing a rise in the food's temperature.
- Infrared (IR).
  IR thermography (temperature monitors, thermographic cameras). All objects emit electromagnetic radiation based on their temperatures. According to the law of black-body radiation, objects near room temperature emit thermal energy as infrared radiation, so variations in the temperature can be detected.
  Night-vision devices. Night-vision devices convert photons (just think of them as particles of light for now) into electrons, which are amplified by a chemical and electrical process and then converted back into visible light. Infrared sources can be used to augment the available light, increasing the visibility in the dark.
  IR heating (IR heat lamps, IR saunas). IR data transmission (remote

controls, optical-fibre communication)

– Visible Light
  Human Vision. Human eyes are only sensitive to a small fraction of the electromagnetic spectrum. The wide variety of colours that we see is actually built up from the relative intensities of red, green and blue light collected by the three colour detectors in our eyes.
– Ultraviolet (UV).
  UV sterilising (drinking water treatment, disinfection of medical facilities, etc.) Short-wavelength UV light can damage the DNA's in living organisms. A microorganism exposed to germicidal UV light might not be able to reproduce, and becomes harmless. For the same reason, overexposure to UV radiation present in the sunlight can cause sunburn, or even skin cancer.
  Fluorescent dyes (black light fluorescent paint, UV watermarks) UV radiation can cause many substances to glow through chemical reactions. UV watermarks that are visible under UV light are used to prevent counterfeiting of currency, or forgery of important documents such as passports and ID cards.
– X-rays.
  Medical imaging (X-ray imaging, CT scans). X-rays are very energetic and thus very penetrating. They can pass through human body easily to form an image giving information about the structures of tissues and bones.
  Security checking (luggage scanners).
  X-ray crystallography. X-rays can be diffracted by the lattice of atoms in a crystal. The diffraction pattern gives information about the structure of the lattice. X-ray crystallography is a very important experimental technique to study the microscopic structures of new materials.
– γ-rays.
  Radiation therapies (cancer treatment). γ-rays have extremely high frequencies. They are even more energetic and penetrating. They can be used to damage the DNA of cancerous tissue, and hence kill the cancerous cells as a treatment.
  Medical imaging (PET scans). Positron emission tomography (PET) uses a radioactive tracer to produce γ-rays within the tissues of interest. The energy and location of these γ-rays can be detected and sent to a computer to build up a 3D image of the body part.

Example 9.5 A beam of electromagnetic radiation is known to have a frequency of 25 THz in vacuum. What is its wavelength?

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8}{25 \times 10^12} \quad \Rightarrow \quad \lambda = 1.2 \times 10^{-5}\ \text{m} \quad \text{(infra-red)}$$

Wave intensity

Energy can be transmitted along a wave, the degree to which this energy is concentrated is called the intensity of the wave. In this case "concentration" has two meanings – concentration in time and concentration in area S[6]

[6] To avoid confusion, I reserved letter 'A' for wave amplitude and chose 'S' to represent an area.

intensity of a wave is defined as the power P per unit area on a cross section S:

$$I = \frac{P}{S}$$

The unit of wave intensity is Watts per meter: $[I] = \text{W m}^{-2}$
[7].

[7] ⚠The intensity is proportional to square of its amplitude: $I \propto A^2$

The intensity of a wave decreases as it spreads out in space, in fact Intensity is a great example of the *inverse square law*. Intensity at distance of r from a point source is inversely proportional to r^2: $I \propto \dfrac{1}{r^2}$ [8]

[8] As a wave produced from a point source travels out by a distance r away from the source, the energy it carries is spread uniformly over the surface area of the sphere of radius r, that is: $S = 4\pi r^2$. So the intensity of a wave obeys an inverse square law: $I \propto \frac{1}{r^2}$.

Example 9.6 If the amplitude of an incoming wave is increased by 50%, what is the increase in the wave intensity?

$I \propto A^2 \quad \Rightarrow \quad \frac{I'}{I} = \left(\frac{A'}{A}\right)^2 = (1 + 50\%)^2 = 2.25 \quad \Rightarrow$ so intensity is increased by 125%

Example 9.7 Two observers A and B are at a distance r_A and r_B from a point source where $r_A = 2r_B$. (a) Find the ratio of their intensities $\frac{I_A}{I_B}$. (b) Find the ratio of their amplitudes $\frac{A_A}{B_B}$.

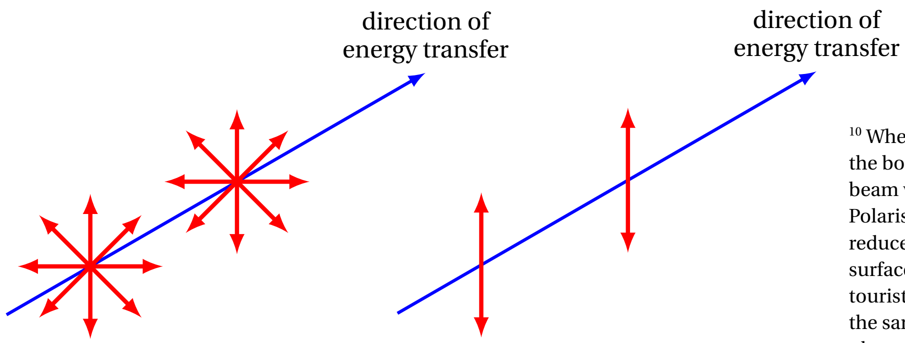
$$I \propto \frac{1}{r^2} \quad \Rightarrow \quad \frac{I_A}{I_B} = \left(\frac{r_B}{r_A}\right)^2 = \left(\frac{1}{2}\right)^2 \quad \Rightarrow \quad \frac{I_A}{I_B} = \frac{1}{4}$$

$$I \propto A^2 \quad \Rightarrow \quad \frac{A_A}{A_B} = \sqrt{\frac{I_A}{I_B}} = \sqrt{\frac{1}{4}} \quad \Rightarrow \quad \frac{A_A}{A_B} = \frac{1}{2}$$

Polarisation

Plane polarisation

A wave is **plane-polarised**, or simply **polarised**, if the vibration is in one single direction at right angle to the direction of propagation of energy



an unpolarised wave (wave vibrations in multiple directions)

a plane-polarised wave (wave vibrations in one direction only)

Polarisation[9] is a phenomenon associated with *transverse* waves only. Longitudinal waves (such as sound waves) cannot be polarised because longitudinal vibrations are always parallel to direction of energy transfer. One of the most important examples are polarisation of *electromagnetic waves*. Sunlight, light emitted from incandescent bulbs etc is unpolarised because they are emitted randomly from their sources, so they contain all planes of polarisation. Lasers, diodes, microwave antennae and a few others produce polarised emissions. A few of the great many applications of polarisation are:

- polarising sunglasses & photography [10]
- polarised 3D films[11]
- liquid-crystal display (LCD) technology[12]
- radio transmission and reception[13]

Polarisers & Malus's law

A plane-polarised light can be produced from unpolarised light using a **polariser**.

A polariser blocks vibrations in all planes except the plane of polarisation. This type of polariser, often called a *linear absorptive polariser*, is basically a synthetic transparent plastic sheet made of certain crystals. The thin sheet contains long-chain organic molecules aligned parallel to each other. When an unpolarised light passes through the polariser, there is a strong absorption of the electric field parallel to the alignment of molecules, so vibration of the electric field is allowed to pass in one direction only.

The direction along which vibration is allowed to pass is called the axis of the polariser

[9] ⚠Apart from plane polarisation where the vibrations are fixed in a single direction, there are also *circular* or *elliptical* polarisation, for which the direction of vibrations *rotate* in a plane as the wave travels.

[10] When unpolarised light is reflected at the boundary of two media, the reflected beam would become partially polarised. Polarising sunglasses use this effect to reduce the sunlight reflected from shiny surfaces, such as a lake (for sightseeing tourists) or road surfaces (for drivers). For the same reason, photographers often place a polarising filter in front of the camera lens to darken skies and increase the contrast.

[11] The eyeglasses worn by viewers contain a pair of polarising filters with mutually perpendicular axes. When two images are projected onto the same screen, each filter restricts the light reaching each eye by passing only one of the images that is polarised in the same direction. This creates an 3D illusion.

[12] Liquid crystal arrays can be realigned by applying an electric field. This causes the axis of polarization to rotate, hence backlight may be allowed to pass or be blocked, forming dark patterns of the display.

[13] Since electric currents flow in certain directions only, so the radio waves and microwaves produced from aerials are intrinsically polarised. This means the reception of signals would be sensitive to a particular direction, but totally insensitive to the normal direction. This explains why altering the orientation of antennas could greatly enhance the quality of reception.

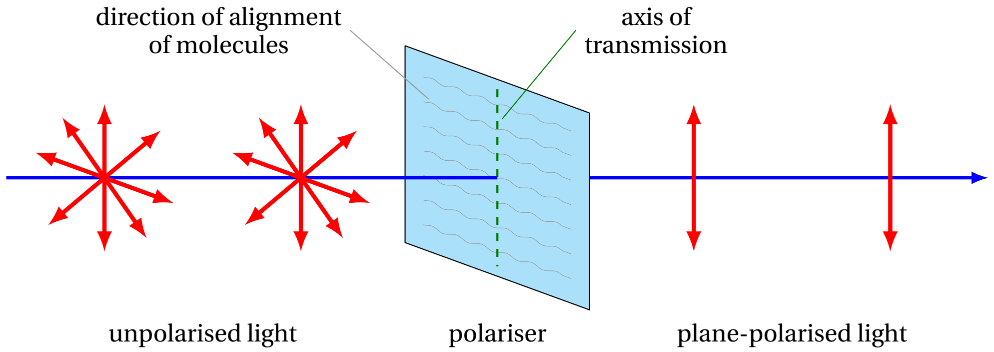
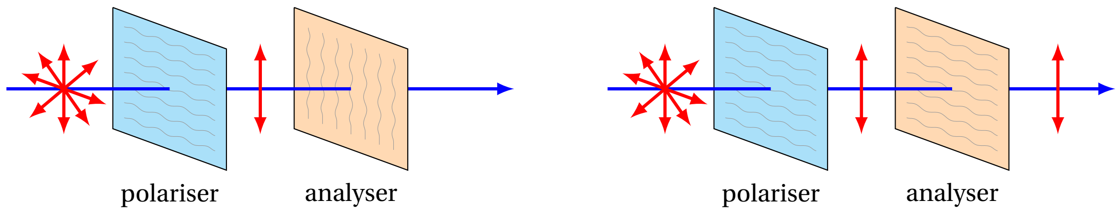


Figure 9.6: unpolarised light goes through a polariser and becomes plane-polarised

Polarised light can be sent through a second polarising filter (sometimes called an *analyser*)



no transmission of light if polariser and analyser are aligned at right angles

polarised light is unaffected if polariser and analyser are aligned in parallel

If polarised light of initial intensity I_0 has an angle θ to the axis of analyser then **Malus's law**[14] states that transmitted intensity is given by:

$$I = I_0 \cos^2\theta$$

[14] ⚠This isn't actually on the spec but it's pretty critical so we often include it.

This is because only electric field parallel to axis of analyser is transmitted, so transmitted amplitude[15] satisfies: $A = A_0 \cos\theta$, where A_0 is the initial amplitude.

[15] This is actually the amplitude of the electric field strength.

Recall that intensity is proportional to square of amplitude, so $I = I_0 \cos^2\theta$

Example 9.8 A beam of light polarised in the vertical direction has an amplitude A and intensity I. It passes through a polarising filter whose axis of polarisation is at 45° to the vertical. (a) What is the amplitude and the direction of polarisation of the emerging beam? (b) If the emerging beam then enters another filter whose axis of polarisation is at 75° to the vertical, what is the amplitude and intensity of the emerging beam?

through first filter: $A_1 = A\cos\theta_1 = A\cos 45° \Rightarrow A_1 = \frac{1}{\sqrt{2}}A$ light is polarised in a direction at 60° to the vertical
through second filter: $A_2 = A_1\cos\theta_2 = \frac{1}{\sqrt{2}}A \times \cos(75° - 45°) \Rightarrow A_2 = \frac{\sqrt{6}}{4}A$
$I_2 = \left(\frac{\sqrt{6}}{4}\right)^2 I \Rightarrow I_2 = \frac{3}{8}I$
or, $I_2 = I\cos^2\theta_1\cos^2\theta_2^2 = I \times \cos^2 45° \times \cos^2(75° - 45°) \Rightarrow I_2 = \frac{3}{8}I$

Doppler effect

Relative motion between wave source and the observer causes a change in observed frequency, this is known as the **Doppler effect**

Any type of wave can exhibit Doppler effect - we are extremely familiar with the changes in pitch as a car drives toward, alongside, then away from us. When wave source moves towards observer, a higher frequency is observed and when wave source moves away from observer, a lower frequency is observed. We can see clearly the direction in which these ducklings are swimming thanks to the same physics:

Figure 9.7: Doppler Ducks...

The Doppler effect also occurs if source is at rest but observer is in motion, as long as there is *radial* motion between source and observer, there is shift in frequency.

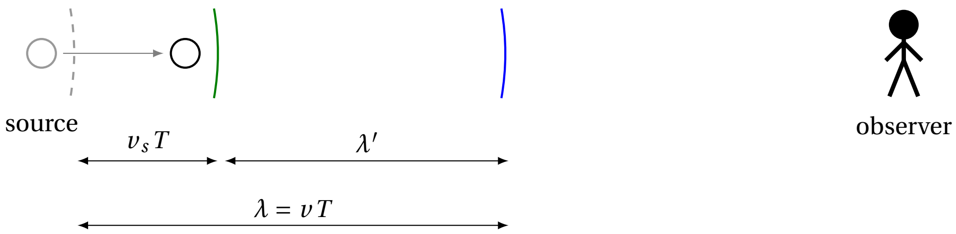
Doppler effect finds its use in many areas, some examples are

- *Doppler radars*: used to measure velocity of moving target (speeding cars, tennis balls, etc.)
- *Doppler ultrasonography*: used to image blood flow in human bodies
- *Astronomy*: used to study motion of stars and galaxies

Explanation of the Doppler effect

Suppose wave source is moving towards the observer at speed v_s, at time $t = 0$, the source emits a wavefront which travels at speed v.

After one period, the wavefront travels forward by a distance of vT. At same time, source moves forward by a distance of $v_s T$ and emits a new wavefront:



If the source is at rest, observer simply perceives a wavelength $\lambda = vT$. If the source is moving closer, a shorter wavelength λ' is perceived since frequency is inversely proportional to wavelength, so higher frequency f' is observed. Similar discussion would follow for the case where source moves away from observer.
The change in observed frequency is due to change in wavelength caused by relative motion - if source moves towards/away from observer, apparent wavelength becomes shorter/longer.

Doppler effect equation

We are now ready to derive an equation for the shift of observed frequency. Quantitatively, we can write: $\lambda' = \lambda \mp v_s T$ ("−"/"+" if source is moving closer/away).
Using $v = \lambda f$ and $f = \frac{1}{T}$, this becomes: $\frac{v}{f'} = \frac{v}{f} \mp \frac{v_s}{f}$. rearranging, one finds observed frequency is given by:

$$f' = f \frac{v}{v \mp v_s}$$

Example 9.9 A police car moving towards you at 16 m s^{-1} sirens at 500 Hz. Given that the speed of sound in air is 340 m s^{-1}, at what frequency do you hear the siren?

$$f' = f \frac{v}{v - v_s} = 500 \times \frac{340}{340 - 16} \quad \Rightarrow \quad f' \approx 525 \text{ Hz}$$

Example 9.10 A star emits an H_α line of wavelength 656 nm. (a) What is the frequency of this H_α line? (b) An observer on earth detects a wavelength of 680 nm, what can we say about motion of the star? (c) Find the relative speed of the star with respect to the earth.

The original frequency of H_α: $f = \frac{c}{\lambda} = \frac{3.00\times10^8}{656\times10^{-9}} \approx 4.57\times10^{14}$ Hz
The observed wavelength is longer (redshift), this means the star is moving away, or receding.
To find speed of star, we can consider observed frequency:
$f' = \frac{c}{\lambda'} = f\frac{c}{c+v_s}$

$$\frac{3.00\times10^8}{680\times10^{-9}} = 4.57\times10^{14}\times\frac{3.00\times10^8}{3.00\times10^8+v_s} \quad\Rightarrow\quad v_s \approx 1.10\times10^7\ \mathrm{m\,s^{-1}}$$

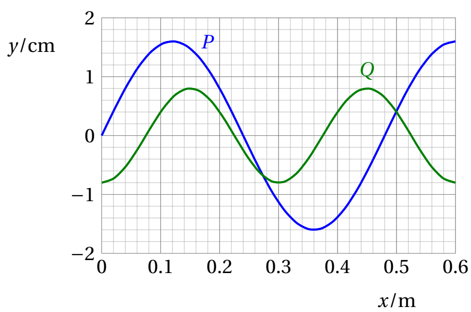
alternatively, we can consider observed wavelength: $\lambda' = \lambda + v_s T$, or: $\Delta\lambda = \lambda' - \lambda = \frac{v_s}{f}$

$$(680-656)\times10^{-9} = \frac{v_s}{4.57\times10^{14}} \quad\Rightarrow\quad v_s \approx 1.10\times10^7\ \mathrm{m\,s^{-1}}$$

End-of-chapter questions

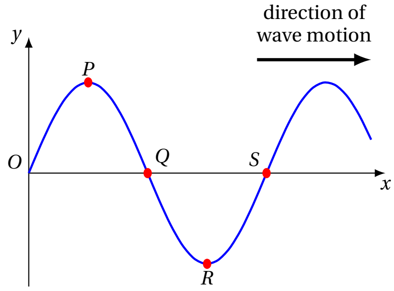
Wave terminologies

Question 9.1 A bottle floating on a water surface is seen to bob up and down over a distance of 20 cm twice each second. What is its amplitude and its frequency?
Question 9.2 A horn produces a note of frequency of 512 Hz. Given that sound travels at 330 m s^{-1} in air, find the wavelength of this sound wave.
Question 9.3 The graph shows the variation of displacement y with the distance x of wave P and wave Q at some instant. (a) For each of the two waves, state the amplitude and the wavelength. (b) Given that both waves have a time period of 0.30 s, find the frequency of the waves, and hence find the speed for each wave.



transverse & longitudinal waves

Question 9.4 An ultrasound waves of frequency 2.0 MHz travels with a speed of 1600 m s^{-1} through water. What is the shortest distance from a point of maximum pressure to a point of minimum pressure?
Question 9.5 The graph shows the variation with distance x of the displacement y of a wave on a stretched rope at time $t = 0$. The wave has a frequency of 1.0 Hz. (a) State and explain whether this wave is transverse or longitudinal. (b) For the points P, Q, R, and S labelled on the graph, which one of them is moving upwards with greatest speed at $t = 0$? (c) Sketch the wave pattern at $t = 0.50$ s. (c) For the points P, Q, R, and S, what are their positions at $t = 0.50$ s?



Question 9.6 Given a slinky toy, suggest how you can demonstrate (a) a transverse wave, (b) a longitudinal wave to your classmates.

Sound waves

Question 9.7 If some alien civilisation blows up the moon one day, can we hear it?

Question 9.8 An oscilloscope is used to measure a sound wave. When the time base is set at 5.0 ms div^{-1}, 3 complete waveforms are observed over 5 divisions. Calculate, (a) the period of the wave, (b) the frequency of the wave.

Question 9.9 Describe the changes to the wave pattern displayed on an oscilloscope if the sound being measured has (a) a higher pitch, (b) a greater volume.

Electromagnetic waves

Question 9.10 Green light has a wavelength of 500 nm in vacuum. What is its frequency?

Question 9.11 An electromagnetic wave has a period of 1.0 ps. (a) What is the wavelength of this wave? (b) What is the number of wavelengths in a distance of 1.0 m?

Question 9.12 (a) What is the frequency of the longest-wavelength ultraviolet wave? (b) What is the frequency of the shortest-wavelength infra-red radiation?

Question 9.13 State a typical value for the wavelength for the following radiation in vacuum: (a) infra-red, (b) X-ray, (c) microwave, (d) ultraviolet.

Question 9.14 A student argues that the speed of an electromagnetic wave is proportional to its frequency because we have $c = \lambda f$. State and explain whether the student's argument is correct.

Question 9.15 (a) Suggest as many properties of electromagnetic waves as you can. (b) Among these properties you have listed, which property of electromagnetic wave is distinct from other transverse waves?

Intensity of waves

Question 9.16 A sound wave in air has an amplitude A_0 and intensity I_0. If the amplitude increases to $3A_0$, what is the new intensity?

Question 9.17 The intensity of radiation arriving normally on a solar panel was 500 W m^{-2}. (a) If the panel has an effective area of 4.0 m^2, how much energy would arrive in one hour? (b) If the radiation has double the amplitude, how much energy would arrive in one hour?

Question 9.18 The intensity I of a sound wave can be given by the formula: $I = kv\rho f^2A^2$, where v is the speed of sound, ρ is the density of the medium, f is the frequency and A is the amplitude. Show that k is a unit-free constant.

Question 9.19 A point source gives out spherical waves. How does the wave amplitude A vary with the distance r from the source?

Polarisation

Question 9.20 A horizontally polarised beam of light of intensity I_0 passes through a polarising filter whose plane of polarisation is at 30° to the horizontal. What is the transmitted intensity?

Question 9.21 When polarized light is sent through some chemical solution, the plane of polarization is rotated. If the intensity is I without the solution being placed in the beam but $0.80I$ after the sugar is placed in the beam, what is the angle of rotation?

Question 9.22 A beam of unpolarised light is sent through two polarising filters A and B. Filter A has its axis of polarisation in the vertical direction, and filter B has its axis of polarisation in the horizontal direction. (a) What is the emergent intensity? (b) A third polarising filter C is inserted between the filters A and B, such that filter C has its plane of polarisation at an angle of $45°$ to the vertical. What is the new emergent intensity?

Question 9.23 Suggest how you may use a stretched elastic rope to demonstrate the phenomenon of polarisation.

Diffraction

Question 9.24 A hill stands between a house and a radio transmitter, but radio signals sent from the transmitter are still received at the house. How can this happen?

Question 9.25 Diffraction can be demonstrated by observing how water waves in a ripple tank go through a gap. Suggest whether the following action would lead to greater or less obvious diffraction: (a) increasing the frequency of the water waves, (b) increasing the amplitude of the waves, (c) increasing the width of the gap, (d) increasing the speed of the waves by increasing the depth of water.

Question 9.26 Microwave ovens use microwaves of frequency of around 2.4 GHz to heat food. The front door of many microwave ovens is made of glass with a metal grid, where gap in the grid are a few mm across. By reference to the wavelength of the microwave, explain how the front door keeps microwaves in but could let light out.

Doppler effect

For the questions below, take the speed of sound to be 340 m s^{-1}.

Question 9.27 A car travels with a constant speed along a straight road. The car's horn is known to sound at a frequency of 500 Hz, but an observer standing on the roadside hears a frequency of 450 Hz. What is the magnitude and the direction of the car's velocity relative to the observer?

Question 9.28 An ambulance travelling at a steady speed of 25 m s^{-1} passes close to a stationary observer. The warning signal on the ambulance has a frequency of 1500 Hz. What is the overall change in the frequency heard by the observer as the ambulance goes by?

Question 9.29 If the motion of the source is at right angles to an observer, state and explain whether there will be a Doppler effect?

Question 9.30 A girl sits on a horizontal platform that is rotating about a vertical axis. The girl moves in a circular path with a constant speed of 8.0 m s^{-1}. The girls starts blowing a whistle which emits a sound of frequency 1200 Hz. To an observer standing at a distance, (a) find the maximum and the minimum frequency heard. (b) Describe the variation in the frequency of the sound heard by the observer.

Question 9.31 A spectral line of a particular wavelength is emitted from a distant star. The wavelength observed is found to vary periodically from a maximum value to a minimum value and back to the maximum value. What can you say about the motion of the star?

Question 9.32 [This question is beyond CAIE syllabus] (a) A source emits sound wave of frequency f. If the observer, rather than the source, is in motion, suggest whether there is any change in the apparent frequency f' heard by the observer. (b) If the observer is moving towards the source at a velocity of v_o, show that the observed frequency f' is: $f' = f\frac{v+v_o}{v}$. (c) If the observer is moving away from the source, derive a similar expression for the observed frequency.

Superposition of Waves

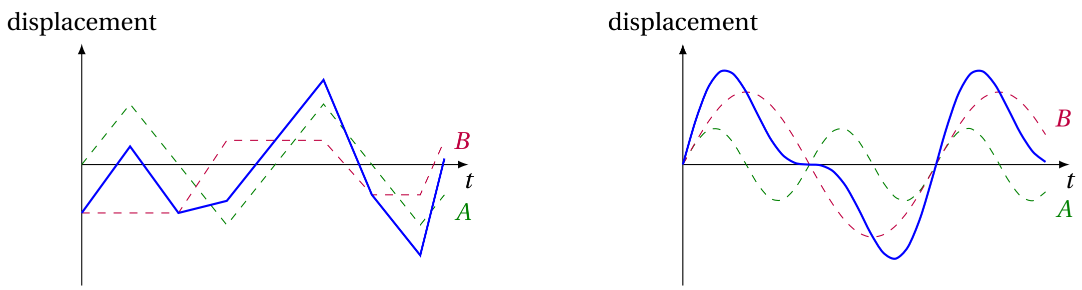
When two of more waves meet together, the resultant motion is a combination of the individuals. They can form a *resultant wave* when they overlap, after which they cross one another and neither is affected. In this chapter, we will study the principle of superposition, and look at two important consequences: the phenomenon of *interference* and *stationary waves.*

Superposition of waves

Principle of superposition

> When two (or more) waves meet to form a resultant wave, the **principle of superposition** states that the resultant displacement is the vector sum of each individual displacement

Example 9.11 Displacement–time graphs for two waves A and B meeting at some point, together with the resultant wave formed at that point.



Two important situations are constructive superposition and destructive superposition

Constructive superposition occurs when resultant wave has greatest possible amplitude. This happens when peaks of the two waves meet together (also trough meets trough)[16] peaks of both waves must arrive with a time difference $\Delta t = 0,\ T, 2T, \cdots$

Destructive superposition occurs when amplitudes of each wave cancel out one another this happens when peak of one wave overlaps with trough of the other wave peaks of both waves must arrive with a time difference $\Delta t = \frac{1}{2}T,\ \frac{3}{2}T,\ \frac{5}{2}T, \cdots$

[16] This statement is implicitly for transverse waves. For longitudinal waves to superpose constructively, the regions of compression (or rarefaction) must overlap.

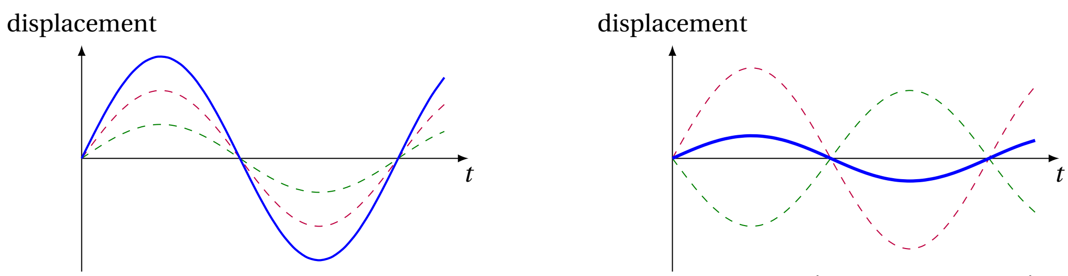


Figure 9.8: constructive and destructive superposition of two transvers waves

Note: longitudinal waves are often represented by transverse diagrams. Transverse are much clearer and easier to read:

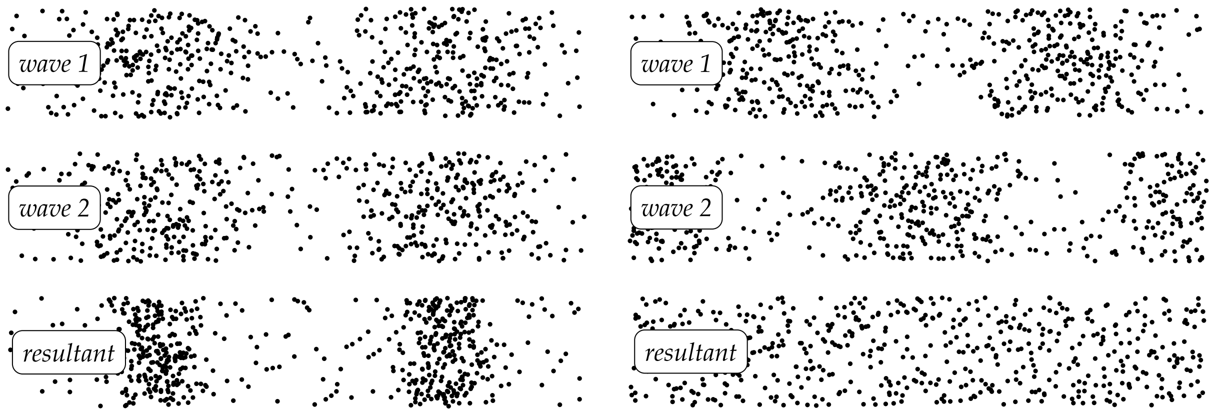


Figure 9.9: constructive and destructive superposition of two sound waves

Path difference

The **path difference** is the difference in distance that two waves must travel from their sources to a given point. It plays a crucial role in wave interference, where waves can either constructively or destructively superpose depending on their path difference. Path difference is generally expressed in multiples of wavelength. To tell whether two waves superpose constructively or destructively:

if $\Delta L = 0, \lambda, 2\lambda, \cdots$, then superposition is constructive
if $\Delta L = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \cdots$, then superposition is destructive

Example 9.12 Two loudspeakers are wired to produce identical sound signals in unison. The sound wave produced has a wavelength of 80 cm. Describe the volume you hear when you are at a distance of (a) 10 m from both speakers, (b) 10 m from one speaker and 12 m from the other?

path difference for case (a): $\Delta L = 10 - 10 = 0$
so constructive superposition, resultant amplitude is large, a loud sound is heard
path difference for case (b): $\Delta L = 12 - 10 = 2.0$ m $\quad\Rightarrow\quad \Delta L = \frac{5}{2}\lambda$
so destructive superposition, resultant amplitude is small, sound is quiet

Phase difference

When two waves or two vibrating particles are compared, it is also useful to describe how much one is out of step with the other in terms of their **phase difference** ($\Delta\phi$) [17]
Phase difference is measured in radians (rad) or degrees (°), if two waves have a phase difference $\Delta\phi = 0, 2\pi, 4\pi, \cdots$, the two waves are said to be **in phase**[18]. In this case, peak of one wave overlaps with peak of the other.
If two waves are not in phase, then they are said to be **out of phase** If $\Delta\phi = \pi, 3\pi, 5\pi, \cdots$, then two waves are completely out of phase, or **anti-phase**[19] in this case, peak of one wave meets the trough of the other.
We can tell whether two waves superpose constructively or destructively from their phase difference as follows:

if $\Delta\phi = 0, 2\pi, 4\pi, \cdots$, then superposition is constructive
if $\Delta\phi = \pi, 3\pi, 5\pi, \cdots$, then superposition is destructive

[17] More logically, we should first define the notion of *phase angle* before talking about the *phase difference*, which is simply the difference between the phase angles of two waves.
Wave motion can generally be described by sine (or cosine) functions. Since the initial displacement of a wave is not necessarily zero, one can introduce a constant term that appears in the argument of the sine function. This term shifts the origin of the sine, and we call that the *phase angle* ϕ of the wave, measured in radians (or degrees). One can think of ϕ as a number giving the fraction in a complete oscillation.
In particular, the displacement y of a transverse wave along the direction x of energy transfer can be given by: $y = A\sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T} + \phi\right)$. You can verify that this indeed describes a sinusoidal wave of wavelength λ and period T travelling at speed $v = \frac{\lambda}{T}$, while ϕ determines when and where the peaks show up.

[18] If $\Delta\phi$ is given in degrees, then two waves are in phase if $\Delta\phi = 0, 360°, 720°, \cdots$

[19] If $\Delta\phi$ is given in degrees, then two waves are anti-phase if $\Delta\phi = 180°, 540°, 900°, \cdots$

Phase difference can also be related to path difference by:

$$\frac{\Delta \phi}{2\pi} = \frac{\Delta L}{\lambda} = \frac{\Delta t}{T}$$

Example 9.13 The diagrams below each shows the displacement of a particular point in two transverse waves. If both waves are travelling in the same direction with a wavelength of 30 cm, what is the phase difference and path difference between each wave?

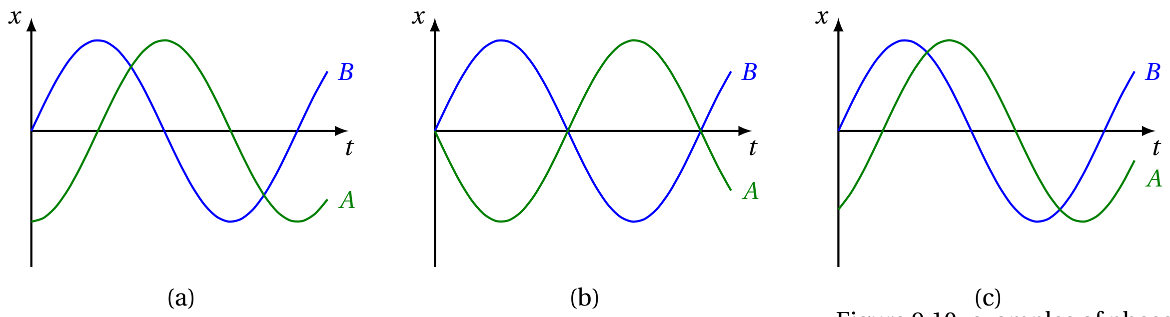


Figure 9.10: examples of phases differences between two waves

(a) $\Delta t = \frac{1}{4}T \Rightarrow \Delta\phi = \frac{1}{4} \times 2\pi = \frac{1}{2}\pi, \quad \Delta L = \frac{1}{4}\lambda = \frac{1}{4} \times 30 =$ 7.5 cm

(b) $\Delta t = \frac{1}{2}T \Rightarrow \Delta\phi = \frac{1}{2} \times 2\pi = \pi, \quad \Delta L = \frac{1}{2}\lambda = \frac{1}{2} \times 30 = 15$ cm

(c) $\Delta t = \frac{1}{6}T \Rightarrow \Delta\phi = \frac{1}{6} \times 2\pi = \frac{1}{3}\pi, \quad \Delta L = \frac{1}{6}\lambda = \frac{1}{6} \times 30 =$ 5.0 cm

Brief summary

The conditions for constructive and destructive superposition can be summarised as follows:

	constructive superposition	destructive superposition
simple criteria	peak meets peak	peak meets trough
time difference	$\Delta t = n \cdot T$	$\Delta t = \left(n + \frac{1}{2}\right) \cdot T$
path difference	$\Delta L = n \cdot \lambda$	$\Delta L = \left(n + \frac{1}{2}\right) \cdot \lambda$
phase difference	$\Delta\phi = n \cdot 2\pi$ or $n \cdot 360°$	$\Delta\phi = \left(n + \frac{1}{2}\right) \cdot 2\pi$ or $\left(n + \frac{1}{2}\right) \cdot 360°$

where n is a whole number: $n = 0, 1, 2, 3 \cdots$
Remember that Δt, ΔL and $\Delta\phi$ are just different ways to describe how much one wave is ahead of/lag behind another wave, so they must be closely interrelated to each other.

Interference

When two (or more) waves of constant phase difference meet, stable regions of constructive and destructive superposition are produced alternately, this phenomenon is called **interference**

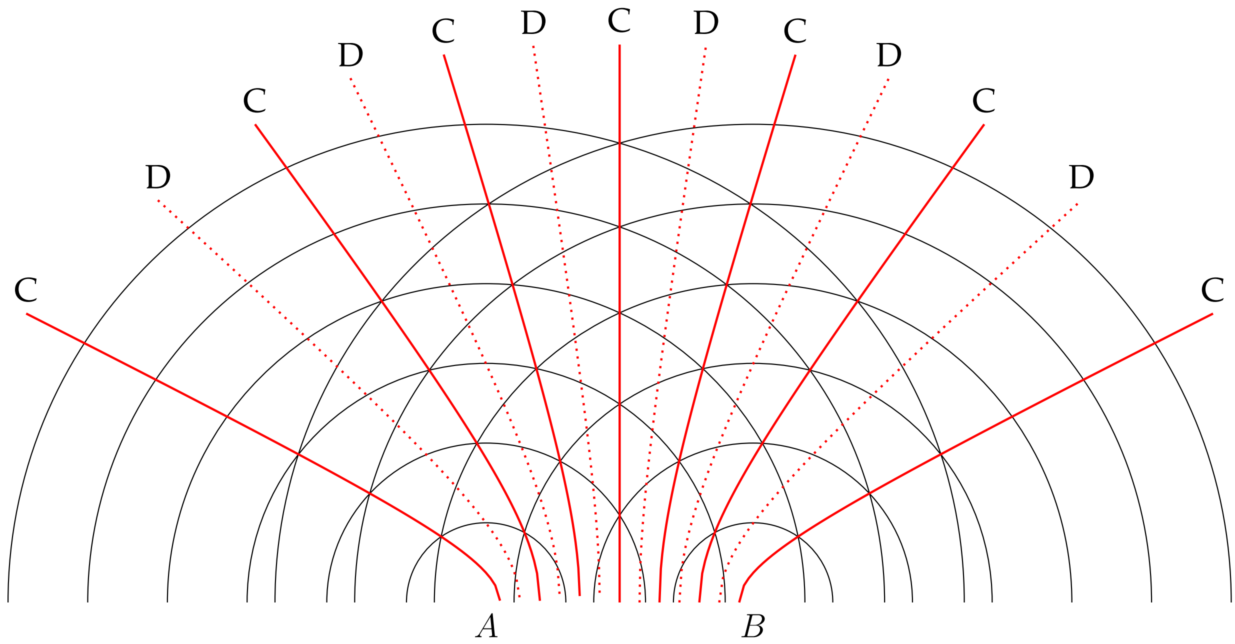


Figure 9.11: constructive (solid lines) and destructive (dashed lines) interference of waves produced from two coherent sources *A* and *B*

Regions of constructive interference are often called *maxima*

- points on line C_0 are of equal distance to wave sources, i.e., path difference $\Delta L = 0$
  so maxima are observed along C_0
- points on C_1, C_2 and C_3 have path difference $\Delta L = \lambda,\ 2\lambda,\ 3\lambda$
  so maxima are also formed along C_1, C_2 and C_3
- equivalently, points on C_n have phase difference $\Delta\phi = n \cdot 2\pi$, where $n = 0, 1, 2, 3, \cdots$

Regions of destructive interference are often called *minima*

- points on D_1, D_2 and D_3 have path difference $\Delta L = \frac{1}{2}\lambda,\ \frac{3}{2}\lambda,\ \frac{5}{2}\lambda$
  so maxima are also formed along D_1, D_2 and D_3
- similarly, points on D_n have phase difference $\Delta\phi = \left(n + \frac{1}{2}\right) \cdot 2\pi$, where $n = 1, 2, 3, \cdots$

Stable interference means regions of maxima/minima remain maxima/minima can be easily seen. This requires **coherent** wave sources, which means constant phase difference over time.
The easiest way to produce coherent waves is to use the the same source, such as

- dippers driven by a common vibrating beam in a ripple tank
- loudspeakers driven by the same signal generator
- a laser beam shone on two slits.

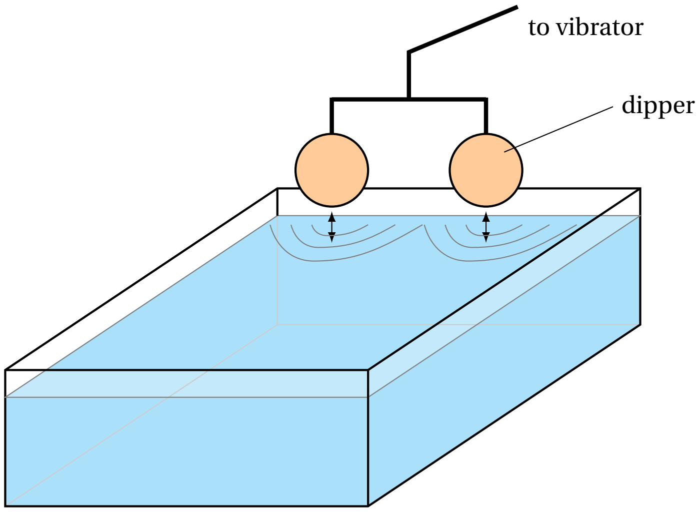


Figure 9.12: Demonstrating interferece of water waves in a ripple tank

Conversely, we do not see interference from beams of light from different lamps or laser sources because light from different sources are usually *incoherent*[20]
This can be overcome by dividing a single beam into several beams using a number of slits. common apparatuses of doing so are the *double-slit* and *diffraction gratings*

[20] Emission of light is associated with the electronic transitions between energy levels in an atom, which is a completely random process. So in general separate light sources are not coherent.

Example 9.14 Two coherent waves meet in space, one of which has an amplitude of 0.30 cm and the other of 0.20 cm. How does the intensity of maxima compare with the intensity of minima?

Amplitude of maxima: $A_{\max} = A_1 + A_2 = 0.30 + 0.20 = 0.50$ cm
Amplitude of minima: $A_{\min} = A_1 - A_2 = 0.30 - 0.20 = 0.10$ cm
Ratio of intensities: $\frac{I_{\max}}{I_{\min}} = \left(\frac{A_{\max}}{A_{\min}}\right)^2 = \left(\frac{0.50}{0.10}\right)^2 \quad \Rightarrow \quad \frac{I_{\max}}{I_{\min}} = 25$

Double-slit interference

Let's take a beam of light being guided through two narrow slits, light waves diffracting through the slits act as coherent wave sources and when they meet on a screen, interference pattern can be seen. Since this experiment is carried out with light, alternating bright and dark fringes are formed.
This is known as *Thomas Young*'s **double-slit experiment** [21]

[21] The nature of light has been argued since the history of human civilisation. It has been long debated whether light is a *wave* or it is made of *particles*. It was until the early 1800s when Thomas Young carried out the famous double-slit experiment that now bears his name, people became certain that light has wavelike properties. We know now of course that it has particle -like properties too. For a detailed review, check out the Wikipedia article: https://en.wikipedia.org/wiki/Light.

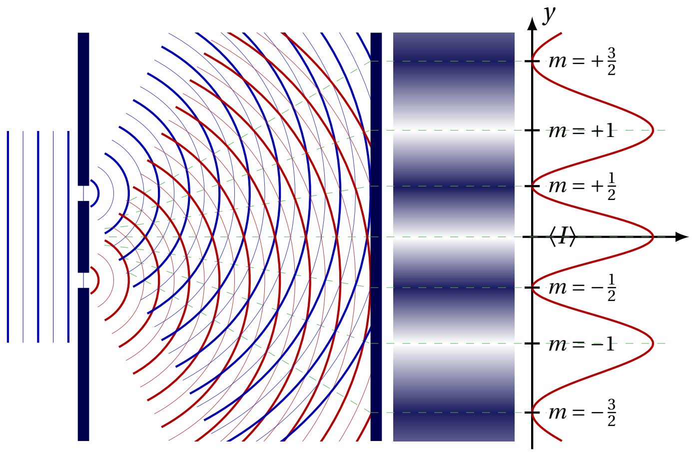


Figure 9.13: Young's double-slit interference experiment

When waves from the two slits meet with path difference $\Delta L = 0, \lambda, 2\lambda, \cdots$, maxima is formed, ie bright fringes are observed.
When waves from the two slits meet with path difference $\Delta L = \frac{1}{2}\lambda, \frac{3}{2}\lambda, \frac{5}{2}\lambda, \cdots$, minima is formed, ie dark fringes are observed.
We can use the geometry of the slit separation, the distance to the screen, and the spacing of the fringes to calculate the wavelength of the light - at least we can if the slit-to-screen distance is much larger than the slit separation, i.e., $D \gg d$.
In this case, bright fringes are nearly equally spaced with a separation of:

$$x = \frac{\lambda D}{d}$$

where d is separation of the two slits, D is slit-to-screen distance
The derivation of this formulae is one you should know and follows:
Consider a point P on the screen where a bright fringe is formed (see figure 9.14)
The path difference between the slits must be a whole number of wavelength:

$$\Delta L = \left|PB - PA\right| = n\lambda$$

Because $D \gg d$, paths PA and PB can be considered approximately parallel, so path difference is length of the segment HB, then we have:

$$\Delta L \approx d \sin\theta$$

In terms of θ, bright fringes are found where:

$$d \sin\theta = n\lambda$$

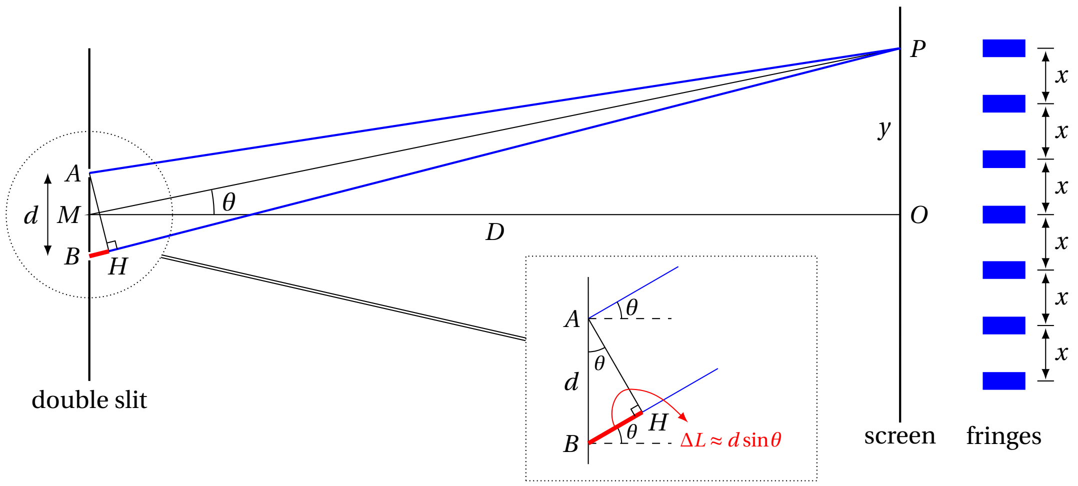


Figure 9.14: Deriving the Young slits formulae.

To convert θ into the coordinate y, we have:

$$\tan\theta = \frac{y}{D}$$

Because θ is very small, the small-angle approximation ($\sin\theta \approx \tan\theta$) can be applied, so positions where the bright fringes show up are:

$$y_n = n \times \frac{\lambda D}{d}$$

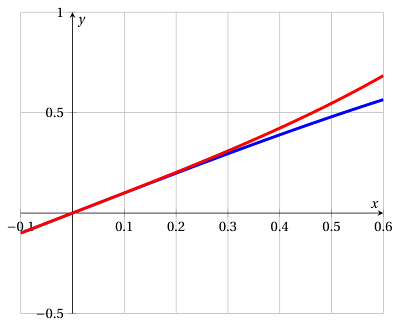


Figure 9.1: For small values of x, tan(x) [red] and sin(x) [blue] can be considered equal.

Here the coordinate subscript n labels the order of the bright fringes. This shows that the $(n+1)$-th fringe is at a fixed distance from the n-th fringe, therefore the separation between neighbouring bright fringes is:

$$x = \frac{\lambda D}{d}$$

Aaltering width of slit could cause a change in *brightness* of fringes.

Example 9.15 A teacher demonstrates the double-slit experiment with a beam of red light produced from a helium-neon laser. The beam of wavelength of 632 nm is sent through a double-slit separated by 0.30 mm and passed onto a wall at about 2.0 m from the slits. (a) What is the fringe separation? (b) If the experiment is carried out using a green laser, what changes do you expect?

Fringe separation: $x = \frac{\lambda D}{d} = \frac{632\times10^{-9}\times2.0}{0.30\times10^{-3}} \quad \Rightarrow \quad x \approx 4.2\times10^{-3}$ m
if replaced by green light, wavelength becomes shorter, so smaller fringe separation

Example 9.16 Coherent light passes through a double slit. Initially, the light intensity from each slit is the same. State the change to the appearance of the fringes if (a) the separation between the slits is increased, (b) The width of one of the slits is reduced.

(a) From $x = \frac{\lambda D}{d}$, shorter λ results in smaller fringe separation
but brightness of the fringes remains unchanged
(b) Reducing width of slit causes amplitude of light from that slit to decrease
at maxima, resultant amplitude decreases, so bright fringes become less bright
at minima, amplitudes do not cancel completely, so dark fringes become a bit brighter
but no change to separation between fringes

Multi-slit interference: diffraction gratings

Passing light through even more slits also produces nice interference patterns, the **diffraction grating** is such an apparatus, with hundreds or thousands of slits per mm

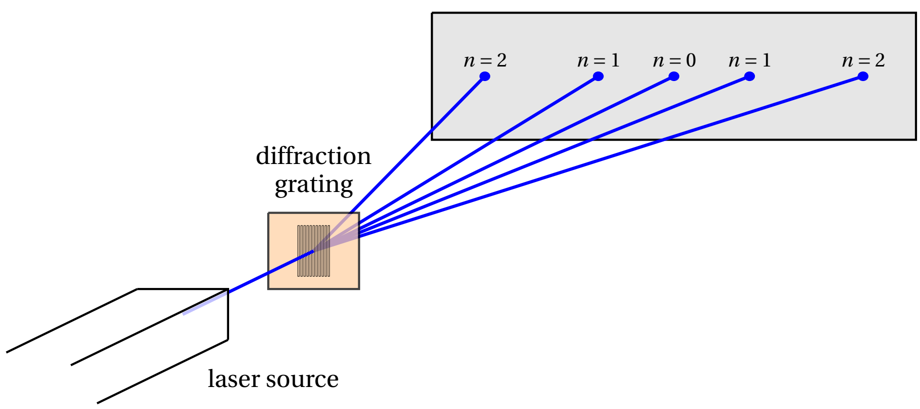


Figure 9.15: multi-slit interference as light passes through a diffraction grating

Compared to double-slit, maxima produced by diffraction grating are *sharper* and *brighter*, maxima are formed when waves from not two but many slits interfere constructively. The location of each maxima from a diffraction grating is given by:

$$d \sin\theta = n\lambda$$

where d is the separation between adjacent slits, and light is incident *normally*.
You do not need to be able to reproduce this derivation, but it is offered here in contrast to the above. For constructive interference, path difference between adjacent slits must satisfy: $\Delta L = n\lambda$. Since slits are so close together, we consider the light rays passing through the slits are nearly parallel. Using the same small angle trick as before, we obtain: $d \sin\theta = n\lambda$ - this is known as the *diffraction grating equation*.
Note that the greatest angle through which a beam of light can be

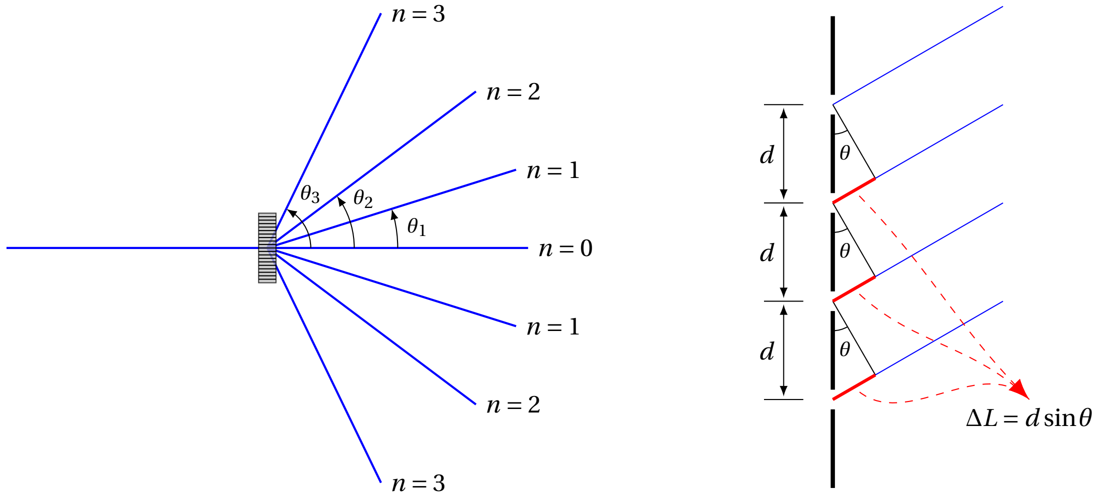


Figure 9.16: Diffraction gratings

diffracted is 90° which puts a constraint on the highest order possible:

$$n_{\text{max}} = \frac{d \sin \theta_{\text{max}}}{\lambda} < \frac{d \sin 90^\circ}{\lambda} \quad \Rightarrow \quad \boxed{n_{\text{max}} < \frac{d}{\lambda}}$$

The value of $\frac{d}{\lambda}$ should be rounded *down* to the nearest whole number to give n_{max}

If white light is passed through diffraction gratings, *dispersion* is observed, this is the cause of the pretty colours seen in a dvd or cd:

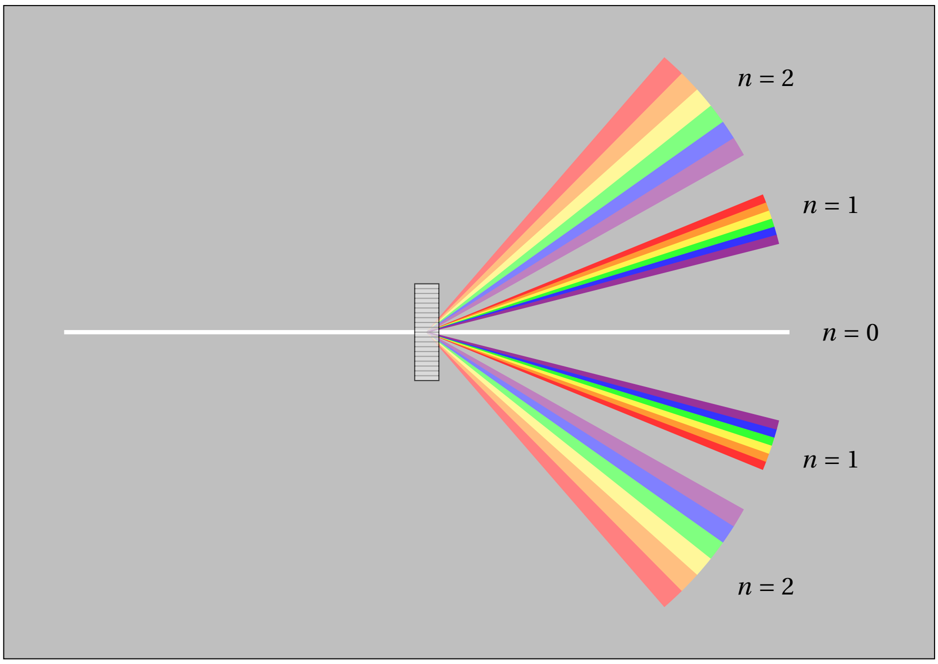


Figure 9.17: dispersion of white light through a diffraction grating

Some notes on the peculiarities of this effect:

- At zeroth order, diffraction grating equation $d\sin\theta_0 = 0\cdot\lambda$ is satisfied at $\theta_0 = 0$ for any λ, so we observe a collection of all colours, i.e., a white central maxima.
- At first order, $d\sin\theta_1 = 1\cdot\lambda$, maxima for long wavelengths appear at greater angles, so the spectrum spreads into a rainbow band, with red light at outer end and violet closer to the centre.
- At higher orders, the spectrum would be more spread out as the frequency dependence of diffracted angle becomes more pronounced. The spectra of different orders may even overlap to give complicated combinations.

The diffraction grating is an import device in analysis of light - it is widely used in measurement of wavelength of light and in telecommunications.

Example 9.17 Light of 632 nm wavelength produces first-order maxima at 16° when passing through a diffraction grating at right angles. How many lines per millimetre are there in the diffraction grating?

slit separation: $d = \frac{n\lambda}{\sin\theta} = \frac{1\times 632\times 10^{-9}}{\sin 16°} \quad\Rightarrow\quad d \approx 2.29\times 10^{-6}$ m
number of lines in 1 mm: $N = \frac{1\text{ mm}}{2.29\times 10^{-6}\text{ m}} = \frac{1\times 10^{-3}}{2.29\times 10^{-6}} \quad\Rightarrow\quad N \approx 436$

Example 9.18 Green light of 510 nm wavelength is incident normally on a diffraction grating with slit separation of 2.0 μm. (a) What is the highest order seen? (b) how many maxima are produced? (c) If blue light is used for the same diffraction grating experiment, suggest the effect on the diffraction pattern.

to find highest order, we have: $n_{\max} < \frac{d\sin 90°}{\lambda} = \frac{2.0\times 10^{-6}}{510\times 10^{-9}} \approx 3.9 \quad\Rightarrow\quad n_{\max} = 3$
1st, 2nd, 3rd-order on either side and 0th-order at centre, so $2\times 3 + 1 = 7$ maxima are formed
blue light has shorter wavelength, so more maxima will be produced with blue light, and separation between each maxima would be closer

Example 9.19 Light of wavelength 630 nm produces a second-order maxima at angle of 60° when it is directed at a diffraction grating. If visible light of another wavelength can also give a maximum at the same angle, what is this wavelength?

using diffraction grating equation: $d \sin\theta = n\lambda$, we find $n\lambda$ is constant for fixed θ
from information given in the question, this constant is $2 \times 630 = 1260$ nm
if $n = 1$, $\lambda = 1260$ nm, which would be infra-red
if $n = 3$, $\lambda = 420$ nm, which would be visible (violet)
if $n \geq 4$, $\lambda \leq 315$ nm, which would be ultraviolet
so the desired wavelength is 420 nm

Diffraction

A wave has the ability to bend around obstacles or pass through narrow gaps

When a wave passes through a aperture/slit/gap/hole or encounters an obstacle, it spreads out/around the corners, this is known as **diffraction** of waves.

Diffraction is a general property of all waves. Some examples of diffraction are:

- sound waves can diffract through an open door
  so you can hear people in the next room talking, even though you cannot see them
- light ray can bend when it goes through/around a small hole/small particles
  sky appears red at sunset as red light is diffracted most by dust particles in atmosphere

Diffraction is a scale-dependent effect - a wave is diffracted most when its wavelength is close to size of aperture/obstacle. Because a single object is close to an "infinite gap", longer wavelengths usually diffracts more around an edge than a wave of shorter wavelength.

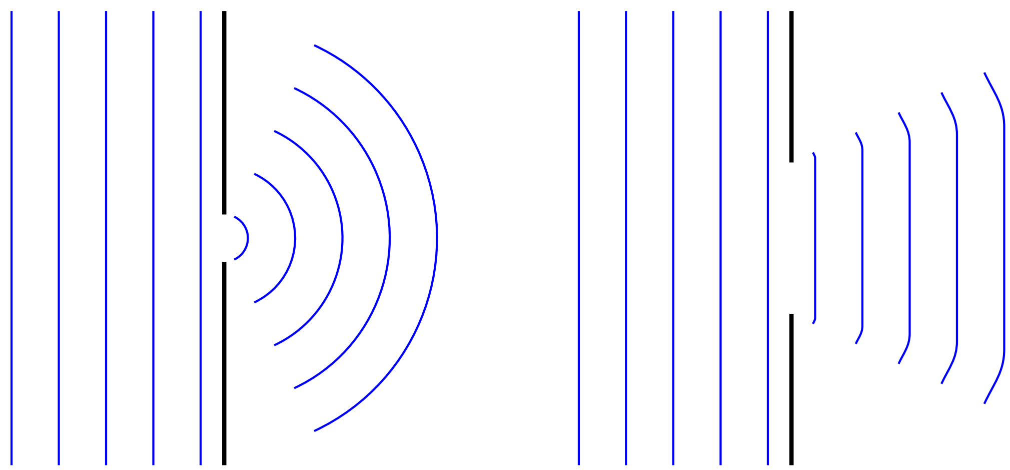


Figure 9.18: diffraction of a wave as it passes through an aperture of width that is (a) close to the wavelength, (b) greater than the wavelength

If we go beyond the GCSE understanding of diffraction as above we can we can see that the light passing through a single slit forms a diffraction pattern somewhat different from those formed by double slits or diffraction gratings, which we discuss earlier in the chapter on interference.

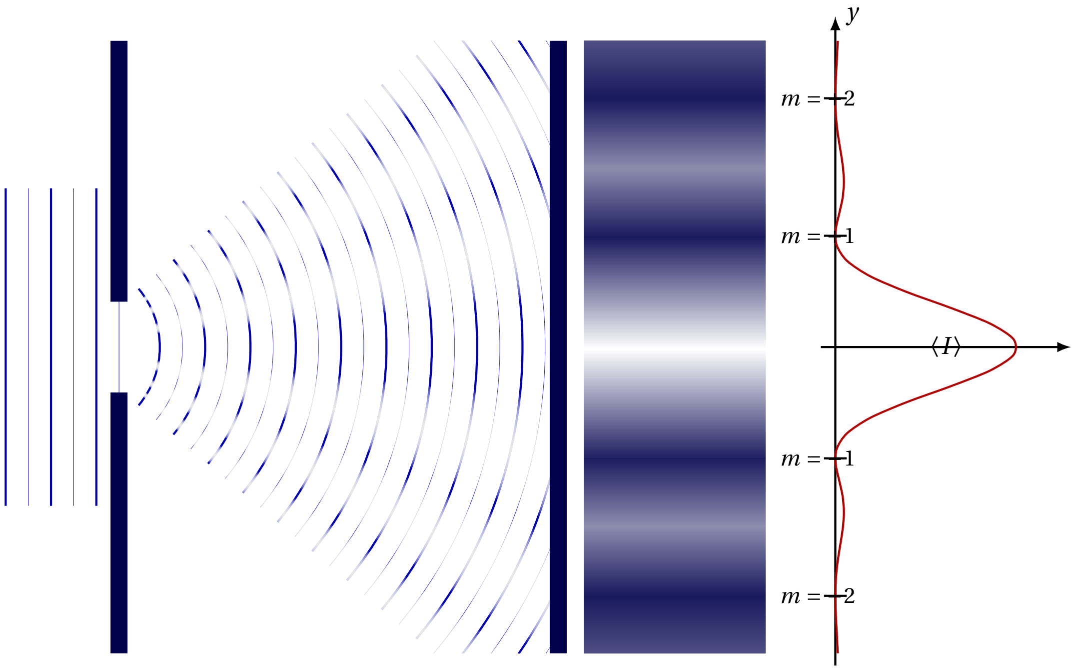


Figure 9.19: Single slit diffraction

Figure 9.19 Shows a single-slit diffraction pattern as we would obseve with eg. a laser. Note that the central maximum is larger than maxima on either side and that the intensity decreases rapidly on either side. In contrast, a diffraction grating produces evenly spaced lines that dim slowly on either side of the center.

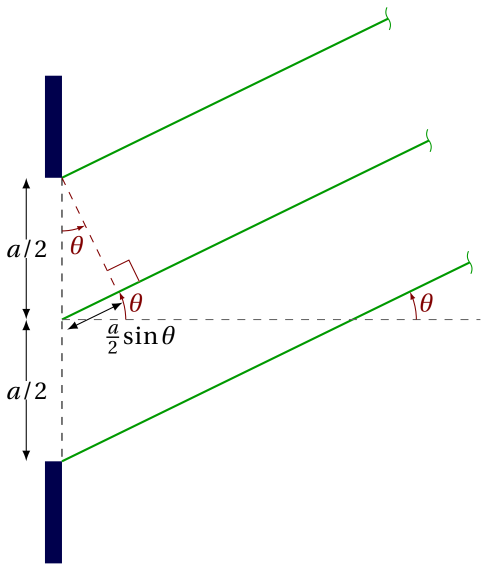


Figure 9.20: Analysis of single slit diffraction

The analysis of single-slit diffraction is illustrated in Figure 9.20 in which the light arrives at the slit, illuminating it uniformly and is in phase across its width. We then consider light propagating onwards from different parts of the same slit.

Assuming the screen is very far away compared with the size of the slit, rays heading toward a common destination are nearly parallel. When they travel straight ahead, they remain in phase, and we observe a central maximum. However, when rays travel at an angle θ relative to the original direction of the beam, each ray travels a different distance to a common location, and they can arrive in or out of phase. The ray from the bottom travels a distance of one wavelength farther than the ray from the top. Thus, a ray from the center travels a distance $\lambda/2$ less than the one at the bottom edge of the slit, arrives out of phase, and interferes destructively. A ray from slightly above the center and one from slightly above the bottom also cancel one another. In fact, each ray from the slit interferes destructively with another ray. In other words, a pair-wise cancellation of all rays results in a dark minimum in intensity at this angle. By symmetry, another minimum occurs at the same angle to the right of the incident direction (toward the bottom of the figure) of the light.
As seen in the figure, the difference in path length for rays from either side of the slit is $a\sin\theta$ and we see that a destructive minimum is obtained when this distance is an integral multiple of the wavelength. Thus, to obtain destructive interference for a single slit,

$$a\sin\theta = m\lambda$$

Where: m is the order of the minimum.
a is the slit width,
λ is the light's wavelength,
θ is the angle relative to the original direction of the light

Stationary waves

Formation of stationary waves

Stationary waves are often formed when a travelling wave is *reflected*. Forward-moving wave and backward-moving reflected wave combine to form stationary wave.

When two waves of same frequency and same speed travel in opposite directions and meet together, they superpose to form stable regions of constructive and destructive interference, this forms a **stationary wave**, also known as a **standing wave**

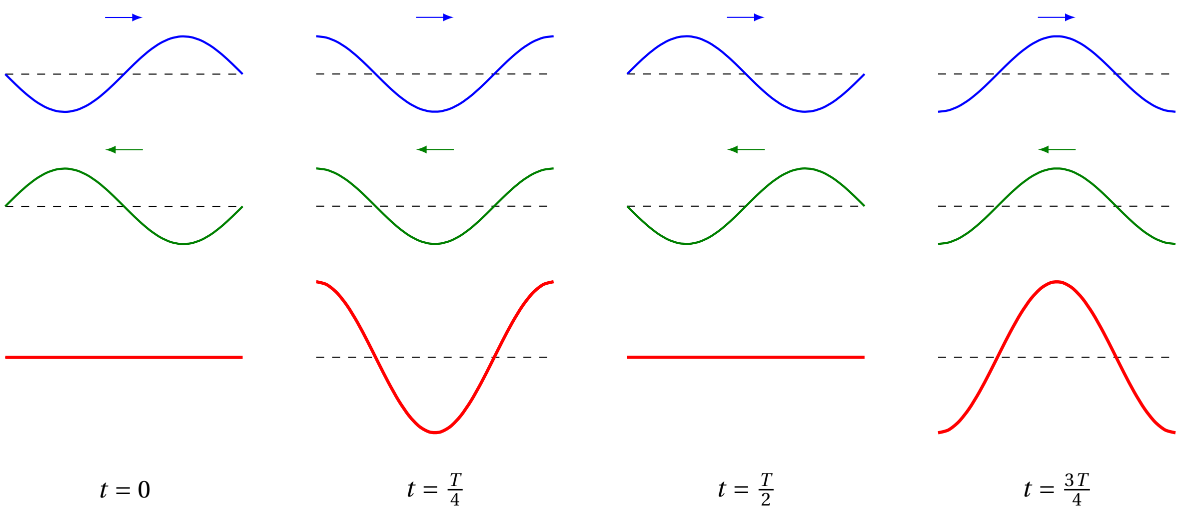


Figure 9.21: The formation of a stationary wave (red) due to the superposition of one wave travelling to the right (blue) and another wave travelling to the left (green)

from this one can visualise how the pattern of a stationary wave varies with time (see Figure 9.22. There are points in the wave where destructive interference always occurs, these points are called **nodes**, which have zero amplitudes.
The points oscillating with the greatest amplitudes are called **anti-nodes**. Points other than nodes and anti-nodes oscillate with different amplitudes.

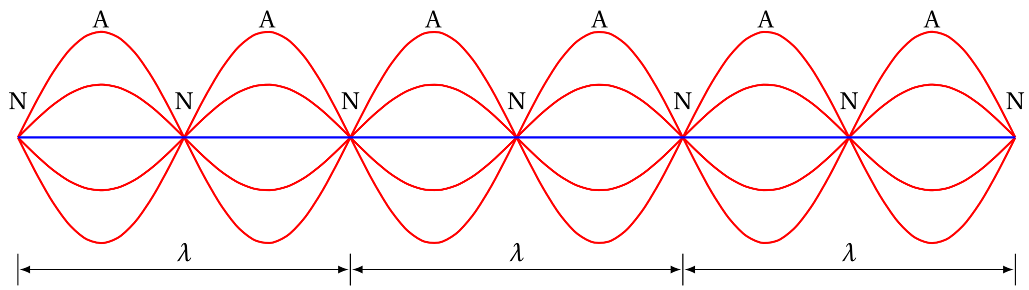


Figure 9.22: variation of a stationary wave pattern with time

The distance between two adjacent nodes is half of one wavelength. This allows us to say::

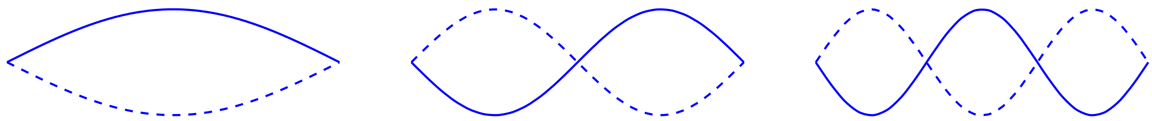
the wavelength $\lambda = 2 \times$ node-to-node distance

Stationary wave pattern & boundary conditions

Patterns of stationary waves depend heavily on the boundary conditions The end can be *fixed*, also called a *closed* end, to form a *node* - such as with a wave on a string.
The end can also be *free* to oscillate, known as an *open* end, to form *an anti-node* - such as you might find in a flute.

Stationary wave between two fixed ends

With both ends fixed, the two ends are both nodes, three patterns with the longest wavelengths and lowest frequencies are shown:



fundamental mode 2nd harmonic 3rd harmonic

The longest-wavelength mode is called the *fundamental mode*, it also has lowest frequency. Modes with higher frequencies are usually referred to as *excited modes*, or *harmonics*[22] in many texts different modes are labelled as 1st, 2nd, 3rd *harmonics*, etc.
If two ends are separated by a distance of L, then wavelengths for different modes are:

$$\lambda_1 = 2L \qquad \lambda_2 = L \qquad \lambda_3 = \frac{2L}{3} \qquad \cdots \qquad \lambda_n = \frac{2L}{n}$$

Making comparison with the fundamental wavelength λ_1 λ_1, we have

$$\lambda_2 = \frac{\lambda_1}{2} \qquad \lambda_3 = \frac{\lambda_1}{3} \qquad \cdots \qquad \lambda_n = \frac{\lambda_1}{n}$$

Since wave speed remains constant as wave travels through the same medium, the frequency of each mode is inversely proportional to its wavelength, hence,

$$f_2 = 2f_1 \qquad f_3 = 3f_1 \qquad \cdots \qquad f_n = nf_1$$

[22] These modes are closely related to the notes played on musical instruments.

Example 9.20 *Melde's experiment* demonstrates a stationary wave formed on a vibrating string using an oscillator set at a particular frequency. Given that the length of the string is 80 cm. (a) What is the wavelength of this wave? (b) If one wants to construct a stationary of longer wavelength, suggest what changes can be made?

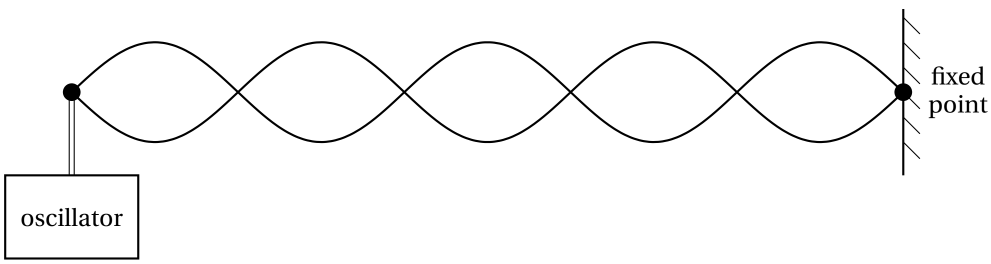


Figure 9.23: Melde's experiment: demonstration of stationary waves on a tense string

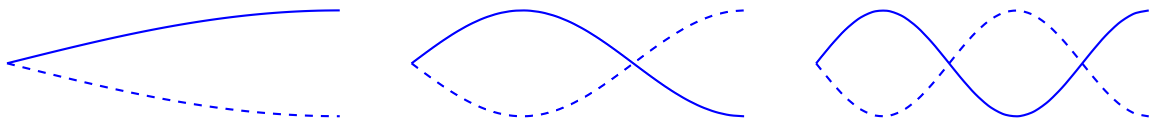
Five loops so five half-wavelengths $\Rightarrow$ $L = \frac{5}{2}\lambda$ $\Rightarrow$ $\lambda = \frac{2}{5}L = \frac{2}{5} \times 80$ $\Rightarrow$ $\lambda = 32$ cm
to have larger wavelength, one could do either of the following: use a string with greater length (while keeping frequency of oscillator unchanged) reduce the frequency of oscillation (constant wave speed, so λ increases)
– increase tension in string (wave speed increases

[23] for same frequency, so λ increases)

[23] The speed v of a progressive wave travelling along a string of length L and mass m can be given by the equation: $v = \sqrt{\frac{TL}{m}}$, where T is the tension in the string.

Stationary wave between one fixed end and one open end

In this case, we have a node at one end and an anti-node at the other end, again we give the three patterns with the longest wavelengths and lowest frequencies.



fundamental mode   2nd harmonic   3rd harmonic

longest-wavelength/lowest-frequency mode is also called the *fundamental mode*
excited modes are called *harmonics* as before
wavelengths for allowed patterns under the boundary conditions are

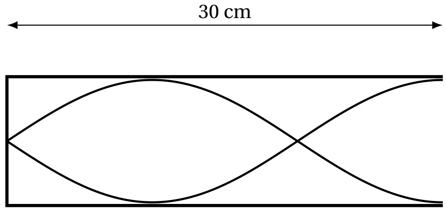
$$\lambda_1 = 4L \qquad \lambda_2 = \frac{4L}{3} \qquad \lambda_3 = \frac{4L}{5} \qquad \cdots \qquad \lambda_n = \frac{4L}{2n-1}$$

comparing with the fundamental wavelength λ_1, we find

$$\lambda_2 = \frac{\lambda_1}{3} \qquad \lambda_3 = \frac{\lambda_1}{5} \qquad \cdots \qquad \lambda_n = \frac{\lambda_1}{2n-1}$$

again the wave speed should be the same for all modes, so the allowed frequencies are

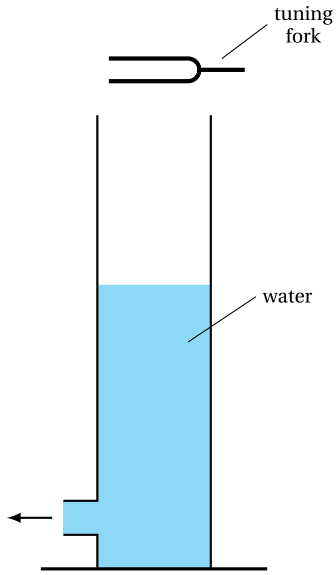
$$f_2 = 3f_1 \qquad f_3 = 5f_1 \qquad \cdots \qquad f_n = (2n-1)f_1$$



Example 9.21 The graph shows a standing sound wave produced in an air column in a closed pipe of length 30 cm. The frequency of this sound wave is 825 Hz. (a) What is the wavelength of the sound wave? (b) Calculate the speed of sound. (c) What is the frequency of the fundamental mode for sound wave in this air column? (d) Describe the motion of an air molecule near the open end.

(a) wavelength of this excited mode: $\lambda = \frac{4}{3}L = \frac{4}{3} \times 30 \times 10^{-2} \quad \Rightarrow$
$\lambda = 0.40$ m
(b) wave speed: $v = \lambda f = 0.40 \times 825 \quad \Rightarrow \quad v = 330 \text{ m s}^{-1}$
(c) wavelength of fundamental mode: $\lambda = 4L = 4 \times 30 \times 10^{-2} \quad \Rightarrow$
$\lambda = 1.2$ m
fundamental frequency: $f = \frac{v}{\lambda} = \frac{330}{1.2} \quad \Rightarrow \quad f = 275$ Hz
(d) note that an anti-node is formed near open end, also sound wave is longitudinal
so air molecule near open end vibrate *horizontally* with greatest amplitude

Example 9.22 A tuning fork is held above a tall cylinder which is initially filled with water. The water level, measured from the bottom of the cylinder, is lowered at a constant rate. A loud sound is first heard when the water level is at 60.0 cm, and the next loud sound is heard when the water level is at 26.8 cm. It is known that the speed of sound in air is 340 m s^{-1}. What is the frequency of the sound wave produced by the tuning fork?



loud sound is heard when stationary wave is set up
node is formed at surface of water
node-to-node distance is $60.0 - 26.8 = 33.2$ cm
wavelength of this wave: $\lambda = 2 \times 33.2 \text{ cm} = 0.664$ m
frequency: $f = \frac{v}{\lambda} = \frac{340}{0.664} \quad \Rightarrow \quad f \approx 512$ Hz

Stationary wave between two open ends

This is left as an exercise for the reader to prove:
You should verify that the allowed wavelengths are: $\lambda = 2L, L, \frac{2}{3}L, \cdots, \frac{2L}{n}, \cdots$
Frequencies of the fundamental and excited modes go as $f = f_1, 2f_1, 3f_1, \cdots, nf_1 \cdots$

Measurement of speed of sound with stationary waves

The required practical version of this is covered in Example 9.22, however there is a version that often appears in exam questions as follows;
To set up a stationary sound wave, we need two travelling waves in opposite directions, this sound wave can be produced by a *loudspeaker* (coupled to a *signal generator*). The forward-travelling wave is reflected at a *metal plate* and the reflected wave superposes with wave from loudspeaker to give a stationary wave. The wave intensity at any point can be detected by a *microphone* and observed on an *oscilloscope*. To verify whether stationary wave is formed, we check whether nodes are formed - the positions of the metal plate and the microphone are adjusted carefully and if there exists a location where no signal is detected, this means microphone is at a node and a stationary wave has been formed between loudspeaker and reflector. To then find wavelength of sound wave, we

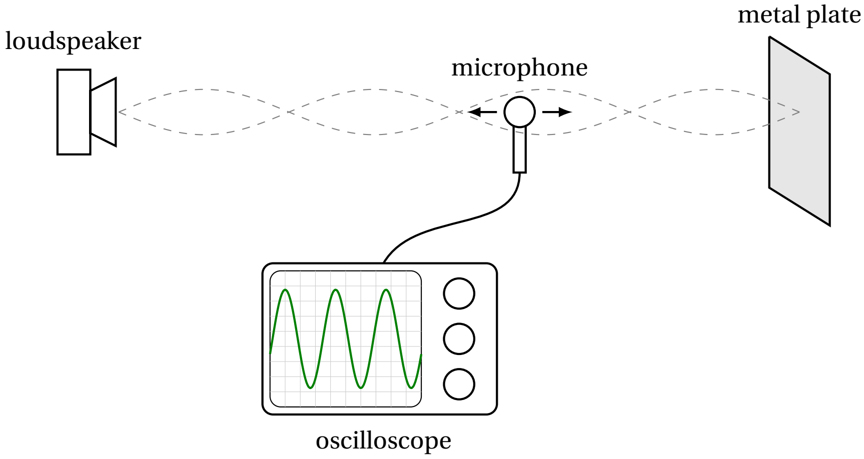


Figure 9.24: arrangement of apparatuses for setting up a stationary sound wave

slowly move microphone to the next node, the wavelength of sound wave: $\lambda = 2 \times$ distance between neighbouring nodes
To find frequency of sound wave, we move microphone to anywhere that is not a node, from time base of the oscilloscope, we can work out period T of the wave. Frequency is then given by the formula: $f = \frac{1}{T}$.
Finally, speed of the sound wave[24] can be calculated: $v = \lambda f$

[24] We should bear in mind that stationary waves do not propagate through space, so rigorously speaking, it is incorrect to say the speed of a stationary way. The wave speed calculated in this way refers to the speed of either of the two travelling waves that give rise to the stationary wave through superposition.

Stationary waves & progressive waves

There are quite a few differences between a stationary wave and a progressive wave:
A progressive wave can transfer energy from one place to another but for stationary waves, vibrational energy is not transferred due to existence of nodes.
In a stationary wave, all part have the same amplitude. Different points in general have different amplitudes over a stationary wave. Nodes have zero amplitude, anti-nodes have greatest amplitudes. Other points have various amplitudes depending on their positions.
In a progressive wave the displacements of each point at a particular moment can be all different, but the largest displacements they can reach are the same (if no loss of energy). This means that the phase difference between any two points in a progressive wave can take any value and different points reach their peaks at different times for a progressive wave so $\Delta\phi$ can be anything from 0 to 2π (or anything from 0 to 360°).
Phase difference between any two points in a stationary wave however, is either 0 or π (or 180°). For points between adjacent nodes, they all reach their highest at same time despite differences in amplitudes, these points are all in phase, so $\Delta\phi = 0$. For points separated by one node, they are anti-phase, because when one is at its peak, the other is at its trough, so $\Delta\phi = \pi$ (or 180°)

End-of-chapter questions

Principle of superposition

Question 9.33 Noise reduction headphones can cancel out external sound waves by producing their own sound waves. What is the phase difference between the external sound wave and the wave produced in the headphone?

Question 9.34 A wave of frequency of 10 Hz travels at a speed of 4.0 m s^{-1}. What is the phase difference between two points 0.50 m apart?

Question 9.35 Two loudspeakers A and B are connected to the same signal generator. They give out sound signals of wavelength 0.50 m. A person stands in a position which is at a distance of 20.0 m from A and 21.0 m from B. Initially, only loudspeaker A is emitting sound. What happens to the intensity of the sound heard by the person when loudspeaker B is also switched on?

Question 9.36 A students thinks that the superposition of waves is about the superposition of the wave amplitudes. Give a counter-example why he is wrong.

Question 9.37 Two separate loudspeakers generate sound waves of slightly different frequencies. (a) Explain why a stable interference is not observed. (b) If an observer stands at a fixed location, state and explain the variation of the intensity of the sound that he hears.

Question 9.38 Two waves A and B are of the same type. Given that they have a phase difference of 120°, and the intensity of wave A is twice that of wave B. On the same graph, sketch the displacement against time to illustrate the two waves.

Question 9.39 Two waves A and B of the same amplitude have a phase difference of 90°. (a) On the same graph, sketch the displacement against time to illustrate the two waves. (b) Sketch the variation of the resultant displacement if A and B superpose together.

Double-slit experiment

Question 9.40 A laser emits a yellow light of 580 nm. It is sent through a set of double slits with a separation of 0.40 mm. An interference pattern is seen on a screen at a distance of 2.5 m from the slits. What is the distance between a bright fringe and its closest dark fringe?

Question 9.41 A double-slit interference pattern is produced on a screen using a red laser of 630 nm wavelength. The screen is 4.0 m from the double slit, and the fringes are seen to have a spacing of 15 mm. What is the separation of the slits?

Question 9.42 Light is incident normally on two narrow slits which are of 2.00 mm apart. A screen is at a distance of 2.5 m from the slits. The distance between the central maximum and the 4^{th} bright spot is found to be 3.2 mm. What is the wavelength of the light?

Question 9.43 In a double-slit experiment, suggest by what factor does the separation between bright fringes change if (a) the distance between the slits and the screen is doubled, (b) the slit separation is doubled, (c) the frequency of incident light is doubled?

Question 9.44 A light of wavelength λ passes through two narrow slits S_1 and S_2 and forms a series of bright and dark fringes on a screen. The n^{th} dark fringe from the central bright fringe is observed at point X on the screen. Find an expression, in terms of λ and n, for $|S_2X - S_1X|$.

Question 9.45 A beam of white light enters a double-slit. Describe and explain the appearance of (a) the central bright fringe, (b) the first bright fringe from the central fringe.

Question 9.46 A beam of light passes through a double slit such that the emergent intensity from each slit is initially the same. Describe the change to the brightness of the fringes if (a) the intensity of light through both slits are increased, (b) the intensity of light through one of the slit is increased.

Diffraction gratings

Question 9.47 A beam of light is incident normally on a diffraction grating that has 1000 lines per mm. The angle between the first order maxima is 60°. Find the wavelength of the light.

Question 9.48 Light of wavelength 590 nm is directed through a diffraction grating. The first order maximum is formed at an angle of 20.0°. (a) What is the slit separation for the diffraction grating? (b) What is the angular separation between the first and second order maxima?

Question 9.49 A beam of light of 480 nm wavelength is incident normally on a diffraction grating which has a slit separation of 3.0 μm. How many intensity maxima can be observed?

Question 9.50 A diffraction pattern is observed when monochromatic light falls on a diffraction grating. If another diffraction grating with twice as many lines per millimetre is used, suggest the possible effects to (a) the total number of diffraction maxima, (b) the angle between the first and second order.

Question 9.51 A beam of light contains two wavelengths 420 nm and 630 nm. A diffraction grating of 6.0×10^5 lines per metre produces a pattern such that the second order of one of these wavelengths overlaps with the third order of the other wavelength. What is the angle θ at which the overlap occurs?

Question 9.52 A diffraction grating is used with several wavelengths of light. The angle θ of the second order maxima is measured for each wavelength λ. If a graph of $\sin\theta$ against λ is plotted, what does the gradient of the graph represent?

Question 9.53 A beam of monochromatic light is incident normally on a diffraction grating. If the grating is rotated about the axis parallel to the incident beam, describe the changes to the positions of diffraction maxima.

Question 9.54 Light of unknown wavelength λ is emitted from a distant star. Suggest the apparatuses you need and what measurements you need take in order to determine λ.

Stationary waves

Question 9.55 How many nodes, including the end points, are there in a standing wave that is three wavelengths long? What about anti-nodes?

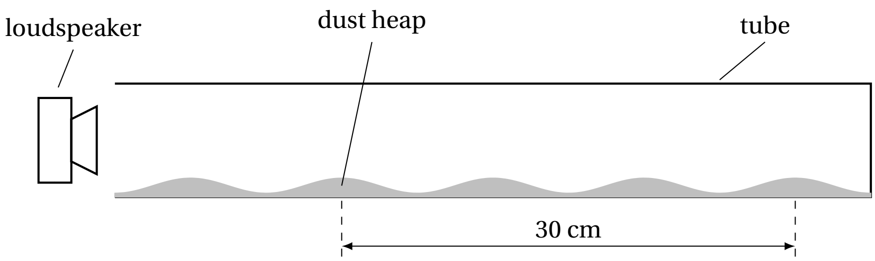
Question 9.56 A stretched rope of length 150 cm is fixed at the two ends. At what wavelengths can stationary waves be set up on this rope?

Question 9.57 (a) A pipe of length l is closed at one end but open at the other. Notes are produced by the pipe if stationary waves are set up. In terms of l and the speed of sound v in the air column, what is the frequency of the lowest note produced in the pipe? (b) If the pipe is open at both ends, what is the lowest frequency?

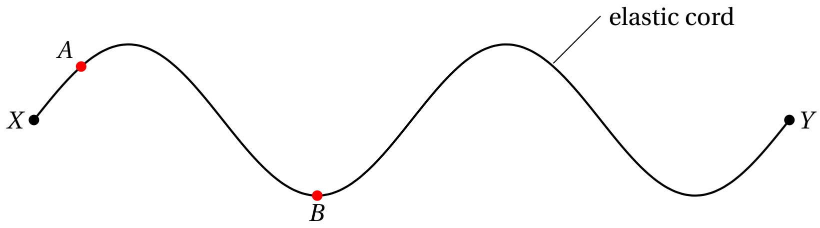
Question 9.58 Suggest and explain how you can lower the pitch of a note on a guitar by altering the length of the string.

Question 9.59 A glass cylinder stands upright and is initially full of water. A loudspeaker emits a sound wave of frequency 850 Hz into the cylinder from above. The speed of sound in air is measured to be 340 m s^{-1}. A tap at the bottom of the cylinder is opened so that the water level slowly decreases. The first loud sound is heard when the water level is at a height of 70.0 cm. When the wave level drops to a height of y cm, a second loud sound is heard. (a) What is the wavelength of the sound wave? (b) Explain why loud sounds are heard for these specific water levels. (c) Find the value of y.

Question 9.60 A glass tube, closed at one end, has a layer of fine dust sprinkled inside on its lower side. A loudspeaker placed near the open end emits sound waves at a particular frequency of 1700 Hz such that a stationary wave is produced inside the tube. The dust is seen to form small heaps of equal spacing as shown below. (a) Explain why heaps of dust are formed. (b) Label the nodes of the stationary wave on the graph. (c) Find the speed of sound in the pipe.



Question 9.61 A stationary wave is set up between the two ends X and Y of an elastic cord. The graph shows the wave pattern at a time $t = t_0$. Given that the frequency of the vibration is 50 Hz. (a) Sketch the wave pattern between X and Y at time $t_1 = t_0 + 10$ ms. (b) Sketch the wave pattern at time $t_2 = t_0 + 25$ ms. (c) State the phase difference between the displacement of point A and that of point B.



10 Cosmology

Stephan's law

An objects luminosity depends on two factors: Its surface temperature Its surface area The relationship between these is known as Stefan's Law or the Stefan-Boltzmann Law, which states: The total energy emitted by a black body per unit area per second is proportional to the fourth power of the absolute temperature of the body
So Stefan's Law shows that the luminosity of a star is directly proportional:

- To its radius $L \propto r^2$
- To its surface area $L \propto 4\pi r^2$
- To its surface absolute temperature $L \propto T^4$

Stefan's Law equation is given by:

$$L = 4\pi r 2\sigma T^4$$

Where: L = luminosity of the star (W)
r = radius of the star
σ = the Stefan-Boltzmann constant
T = surface temperature of the star (K)
The surface area of a star (or other spherical object) can be calculated using: $A = 4\pi r^2$
Where r = radius of the star
Example 10.1 The surface temperature of Proxima Centuri, the nearest star to Earth, is 3000K and its luminosity is 6.506×10^{23}W.
Calculate the radius of Proxima Centuri in kilometres and show your working clearly.

List the known quantities:
Surface temperature, $T = 3000K$
Luminosity, $L = 6.506 \times 10^{23}$W
Stefan's constant, $\sigma = 5.67 \times 10^{-8} W m^{-2} K^{-4}$
Write down Stefan's Law & sub in values
The radius of Proximal Centuri is 106 200 km (4 s.f.)

Wein's law

Black Body Radiator

An ideal black body radiator is one that absorbs and emits all wavelengths.
A black body is a theoretical object, however, stars are the best approximation there is.
The radiation emitted from a black body has a characteristic spectrum that is determined by the temperature alone.

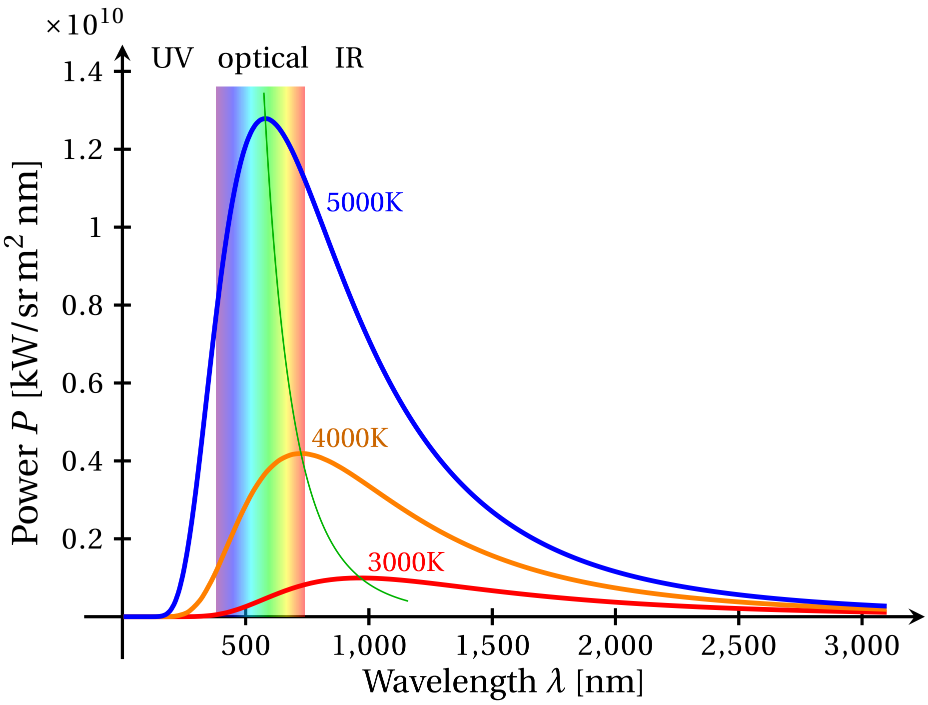


Figure 10.1: Different temperature stars and their spectral output

The intensity-wavelength graph shows how thermodynamic temperature links to the peak wavelength for three different bodies.

Wien's Displacement Law

Wien's displacement law relates the observed wavelength of light from an object to its surface temperature, it states: The black body radiation curve for different temperatures peaks at a wavelength that is inversely proportional to the temperature
This relation can be written as:

$$\lambda_{\text{max}} = \frac{W}{T}$$

Where:
λ_{max} = the maximum wavelength emitted by an object at the peak intensity (m)
T = the surface temperature of an object (K)
Wien's constant is $W = 2.9 \times 10^{-3} mK$
This equation shows that the higher the temperature of a body, the

shorter the wavelength at the peak intensity. Hotter objects tend to be white or blue, and cooler objects tend to be red or yellow.//

Example 10.2 The spectrum of the star Rigel in the constellation of Orion peaks at a wavelength of 263 nm, while the spectrum of the star Betelgeuse peaks at a wavelength of 828 nm. Determine which of these two stars, Betelgeuse or Rigel, is cooler.

List the known quantities Maximum emission wavelength of Rigel = 263 nm = 263 × 10-9 m Maximum emission wavelength of Betelgeuse = 828 × 10-9 m

$$T = \frac{2.9e-3}{\lambda_{max}}$$

Calculate the surface temperature of each star:

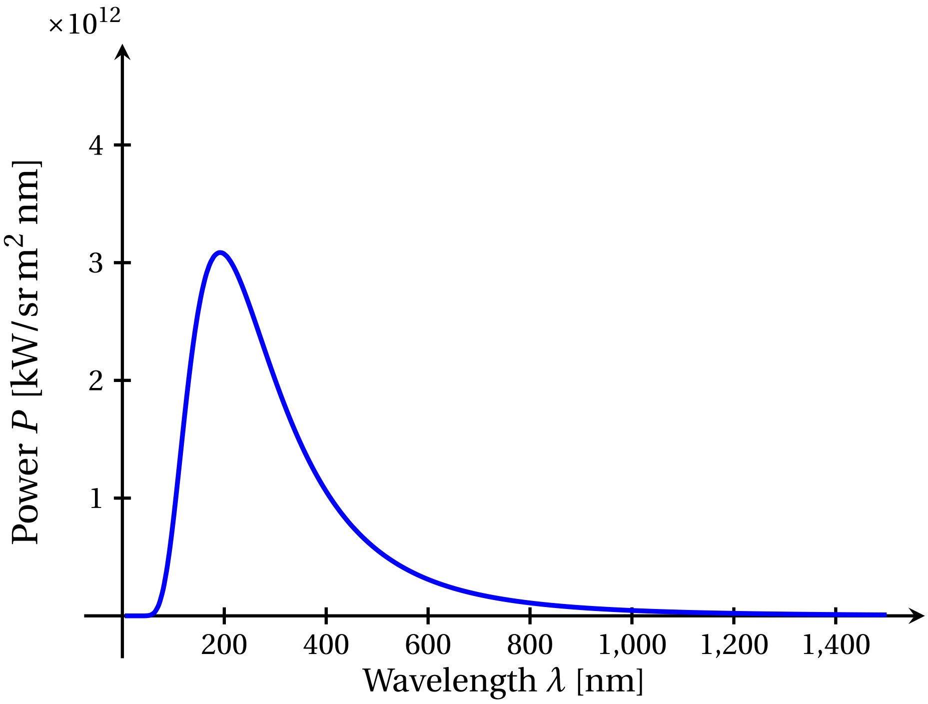
$$\text{Rigel: } T = \frac{2.9e{-}3}{263e{-}9} = 11026 = 11000K$$

$$\text{Betelgeuse: } T = \frac{2.9e{-}3}{828e{-}9} = 3502 = 3500K$$

Step 4: Write a concluding sentence
Betelgeuse has a surface temperature of 3500 K, therefore, it is much cooler than Rigel

End-of-chapter questions

One of the hottest stard known is HD93129A in the Carina nebula. It's continuous spectrum is shown;



11 Electrical Quantities & Components

Electrical quantities

Electric charges

Electric charge is the property of matter that causes it to experience a force in an electric field. Electric charges come in two types, *positive* charges (+ve) and *negative* charges (-ve) [1]. You should be familiar with the fact that like charges repel each other, and opposite charges attract. The interaction between electric charges plays a central role in all electrostatic, chemical and electrical phenomena.

[1] Important to realise that the +ve and -ve signs are to distinguish each type. A -ve charge is not an deficit of charge in the way that a -ve bank account is a debt.

Electric charges are measured in coulombs: $[Q] = \text{C}$
All matter is made of atoms, which consist of a nucleus (+ve) and electrons (-ve) and each electron has a charge of $-e$, where $e = 1.60 \times 10^{-19}$ C is the **elementary charge**.
Typically a body becomes electrically-charged by losing or gaining electrons.

- an object that gains electrons from elsewhere becomes negatively-charged
- an object that loses electrons becomes positively-charged

The charge of any object must be an integer multiple of the elementary charge: $Q = Ne$ This is the same quantity as the charge on an electron or proton.
We say the charge of an object is **quantised**. In the universe, there are a number of conserved quantities - This law of conservation, known as the **conservation of electric charges** appears to be a very significant one. An unbalanced charge cannot be created nor be destroyed - ie you can cause a positron and electron to exist, as the total created charge is zero.
Example 11.1 A metal sphere carries a net charge of $+2.4 \times 10^{-9}$ C. Suggest whether the sphere has gained additional electrons or lost electrons, and also calculate how many electrons has the sphere gained or lost?

> The sphere is positively-charged, so it has lost some of its electrons.
> The number of lost electrons: $N = \frac{Q}{e} = \frac{2.4\times10^{-9}}{1.60\times10^{-19}} \quad \Rightarrow \quad N = 1.5 \times 10^{10}$

Electric current

A separation of charges gives rise to a force. Forces cause accelerations so it's inevitable that some carges move. The flow of electric charges gives rise to electric currents.

> **Electric current** I is defined as the amount of charge flow Q per unit time:
>
> $$I = \frac{\Delta Q}{\Delta t}$$

The unit of electric current: $[I] = \text{A}$ (ampere)
The Ampere is one of the S.I. base units[2]
Though it causes some frustration - the direction of a current is defined as the direction of flow of positive charges. Of course, the current flowing in a conductor is usually due to motion of negatively-charged electrons so **conventional current** is in opposite direction to the electron flow.
In many cases we might use this equation to find charge flow from a current, we can use $Q = It$ only if there is a constant current. For capacitors, AC circuits and in other non-constant circumstances we can use the area under an I-t graph to give the total charge transferred. It can sometimes be useful to find the average current and then proceed with $Q = \bar{I}t$.
Note that current in a circuit can be measured with an **ammeter**:
Example 11.2 A current of 25 mA flows in a wire. How many electrons have passed a point in the wire in one hour?

> Charge flow: $Q = It = 25 \times 10^{-3} \times 3600 \quad \Rightarrow \quad Q = 90\ \text{C}$
> Number of electrons: $N = \frac{Q}{e} = \frac{90}{1.60 \times 10^{-19}} \quad \Rightarrow \quad N \approx 5.6 \times 10^{20}$

Example 11.3 The current in a lamp is increased uniformly from 2.0 A to 8.0 A over 3.0 s. What is the charge flow?

> Average current during this time is: $\bar{I} = \frac{2.0+8.0}{2} = 5.0\ \text{A}$
> charge flow is then: $Q = \bar{I}t = 5.0 \times 3.0 = 15\ \text{C}$
> we can also use area under I-t graph
>
> $$Q = \frac{1}{2} \times (2.0 + 8.0) \times 3.0 = 15\ \text{C}$$

[2] As of the 2019 redefinition of the SI base units, the ampere is defined by fixing the elementary charge e to be exactly $1.602176634 \times 10^{-19}$ C, which means an ampere is an electric current equivalent to 10^{19} elementary charges moving every 1.602176634 seconds. Previously it was defined as the following: when two parallel infinitely long straight wires separated by a distance of 1 m carry two equal currents such that the force per metre between them is 2.0×10^{-7}N, then this current is of 1 A. - It was oddly circular that the notion of electric current was defined based on electric charge, and the unit of charge was defined based on the unit of current: 1 C = 1 A s.

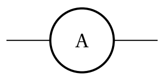
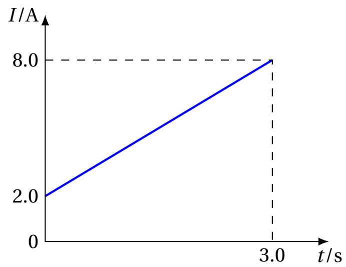


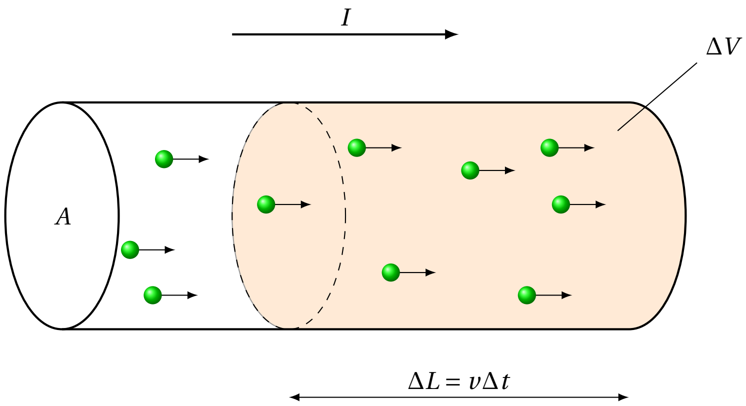
Figure 11.1: The symbol for an Ammeter



Microscopic view of electric currents

The flow of electric current is essentially due to motion of charge carriers due to the electrostatic attraction and repulsion.[3] From an intuitive perspective, current would depend on number of charge carriers, and how fast they move. We'll see that this isn't far from correct. Let's take a wire of cross section A, and see what follows in thinking about current.

[3] In most metallic conductors, charge carriers are those *free electrons*. Charge carriers can also be thought about as positively-charged *holes* (the absence of electrons) in some semi-conductors or *ions* in chemical solutions.



We know current is transfer of charge so let's define n as the density of charge carriers for the material - in other words the number of charge carriers per unit volume: $n = \frac{\text{Number of free charges}}{\text{Volume}}$
The total number N of charge carriers in a volume V can then be given by: $N = nV$
If each carrier has a charge of q and moves with a speed v, then they contribute to a current:

$$I = \frac{\Delta Q}{\Delta t} = \frac{\Delta N q}{\Delta t} = \frac{n \Delta V q}{\Delta t} = \frac{n A \Delta L q}{\Delta t} = \frac{n A v \Delta t q}{\Delta t} \Rightarrow$$

$$\boxed{I = nAvq}$$

In metallic conductors, free electrons act as charge carriers, the equation becomes: $I = nAve$ where e is the charge on an electron.
As with many statistical physics phenomena, the charge carriers have a distribution of speeds as they move so it's more proper to say that v is actually an average velocity, called the **mean drift velocity**
The charge density n is a property of the material in question - ie it's value depends on type of material, good conductors have large n, while semiconductors have smaller n. [4]

[4] We're past the point of considering the world in binary terms of *conductor* and *insulator*, though we still use those terms as shorthand for the extremes of range.

Example 11.4 A length of silver wire of a diameter of 1.2 mm carries a current of 2.0 A. The electron number density in silver is 5.9×10^{28} m^{-3}. What is the mean drift velocity of the electrons?

$$v = \frac{I}{nAe} = \frac{I}{n \cdot \frac{1}{4}\pi d^2 \cdot e} \Rightarrow$$

$$\frac{2.0}{5.9 \times 10^{28} \times \frac{1}{4}\pi \times (1.2 \times 10^{-3})^2 \times 1.60 \times 10^{-19}} \approx 1.9 \times 10^{-4} \text{ m s}^{-1}$$

This calculation shows that each free electron actually travel at very low speeds in a wire

Example 11.5 A metal conductor and a piece of semi-conductor of the same cross section are connected in series. When an electric current is driven through them, compare the mean drift speed of the charge carriers in the two materials.

The same current and same cross section, so mean drift speed $v = \frac{I}{nAe} \propto \frac{1}{n}$
The metal has larger n, so free electrons move at relatively lower speeds
The semi-conductor has smaller n, so charge carriers move at relatively higher speeds

P.d. & e.m.f.

Potential difference and e.m.f.[5] are both defined as the energy transfer per unit charge, both terms are often lumped together as *voltages* in less formal contexts.

[5] e.m.f. stands for *electromotive force*, which is a miserably misleading term due to historical reasons. It has nothing to do with a force, or motivation for that matter. It is better to simply use the abbreviation e.m.f.

The **P.d. (potential difference)** between two points is the energy converted from electrical potential energy to some other form per coulomb of charge flowing from one point to the other.:

$$V = \frac{W}{Q}$$

The e.m.f. (electromotive force) of a power supply is the energy converted from some other form (e.g. chemical) to electrical potential energy per coulomb of charge flowing through the source:

$$\mathscr{E} = \frac{W}{Q}$$

The unit of measurement for p.d./e.m.f.: $[V] = [\mathscr{E}] = \text{V}$ (volt), where $1\,\text{V} = 1\,\text{J}\,\text{C}^{-1}$
The p.d. can tell you the direction of current flow through a component, current always flows from a higher potential to a lower potential.
We measure the p.d. across a component can be measured with a **voltmeter**.

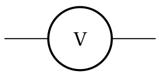


Figure 11.2: The symbol for a Voltmeter

Example 11.6 How much electrical energy is transformed into thermal energy when a charge of 10 C flows through a heater whose p.d. across is 25 V?

$$V = \frac{W}{Q} \quad \Rightarrow \quad W = VQ = 25 \times 10 = 250\,\text{J}$$

Example 11.7 A fully-charged 12 V car battery can supply a total electrical energy of 0.90 MJ. The starter motor of the car requires an average current of 150 A for a period of 2.0s. The battery is not able to recharge due to a fault. How many times can the starter motor be used?

total charge that can be supplied: $Q_{\text{total}} = \frac{W}{\mathcal{E}} = \frac{0.90\times10^6}{12} = 75000$ C
charge needed for each start: $Q = It = 150 \times 2.0 = 300$ C
number of starts possible: $N = \frac{Q_{\text{total}}}{Q} = \frac{75000}{300} = 250$

Example 11.8 Electrons beams are accelerated between a heated cathode and an anode with a potential difference of 40 V. What is the speed of the electrons when they reach the anode?

electrical potential energy is transformed into kinetic energy of electrons:

$$qV = \frac{1}{2}mv^2 \quad \Rightarrow$$

$$v = \sqrt{\frac{2qV}{m}} = \sqrt{\frac{2 \times 1.60 \times 10^{-19} \times 40}{9.11 \times 10^{-31}}} \approx 3.75 \times 10^6 \text{ m s}^{-1}$$

Resistance

The **Resistance** R of an electrical component is defined as the potential difference across divided by the current flowing through it:

$$R = \frac{V}{I}$$

The unit of resistance: $[R] = \text{V A}^{-1} = \Omega$ (ohm) named after Georg Ohm[6].
When a p.d. of 1 V is applied and current flow is 1 A, then resistance is 1 Ω
Note that:

- Resistance of a component does not depend on p.d. applied or current through it.
- Resistance R of a component mainly depends on the following factors:
  - R is proportional to length L of the component: $R \propto L$.
  - R is inversely proportional to cross-sectional area A : $R \propto \frac{1}{A}$.
  - R depends on the material of the component.

this can be written as:

$$R = \rho\frac{L}{A}$$

The constant ρ is called the **resistivity** of the material, a typical value of resistivity for a conducting metal is approx. $\rho \sim 10^{-8}\ \Omega$ m.

[6] Ohm's law was probably the most important of the early quantitative descriptions of the physics of electricity. We consider it almost obvious today. When Ohm first published his work, this was not the case; critics reacted to his treatment of the subject with hostility. They called his work a "web of naked fancies" and the Minister of Education proclaimed that "a professor who preached such heresies was unworthy to teach science." The prevailing scientific philosophy in Germany at the time asserted that experiments need not be performed to develop an understanding of nature because nature is so well ordered, and that scientific truths may be deduced through reasoning alone...

Example 11.9 A current of 0.10 A is driven through a heater when connected to a supply voltage of 220 V. (a) Find the resistance of the heater. (b) If the heater is made from a wire of resistivity $1.8 \times 10^{-5}\ \Omega$ m and a diameter of 0.24 mm, find the length of the wire.

$$R = \frac{V}{I} = \frac{220}{0.10} \quad \Rightarrow \quad R = 2200\ \Omega$$

$$R = \frac{\rho L}{A} \quad \Rightarrow \quad L = \frac{RA}{\rho} = \frac{2200 \times \pi \times (0.12 \times 10^{-3})^2}{1.8 \times 10^{-5}} \quad \Rightarrow \quad L \approx 5.5\ \text{m}$$

Example 11.10 A uniform wire of resistance R has a length of L and a cross-sectional area of A. If the wire is stretched to twice the length while its volume remains constant, what is the new resistance R' in terms of R?

volume: $V = LA =$ constant, so $A \propto \frac{1}{L}$, hence doubling L means A would become halved

resistance: $R = \frac{\rho L}{A} \propto \frac{L}{A} \quad \Rightarrow \quad \frac{R'}{R} = \frac{L'}{L} \times \frac{A}{A'} = 2 \times 2 \quad \Rightarrow \quad R' = 4R$

Electrical power

Current flowing through any electrical component causes energy to be transferred from source to the compnent. The rate at which electrical energy is transferred - the electrical power - is computed as:

$$P = \frac{\Delta W}{\Delta t} \xlongequal{V = \frac{W}{Q}} \frac{\Delta Q \cdot V}{\Delta t} \xRightarrow{I = \frac{\Delta Q}{\Delta t}}$$

$$\boxed{P = IV}$$

Substituting $V = IR$, the formula for electrical power also takes two useful forms:

$$\boxed{P = I^2 R \quad \text{or} \quad P = \frac{V^2}{R}}$$

From these we can see why the national grid, and it's step-up and step-down transformers is such a big part of GCSE - You can see that a very small change in the current flowing can have a large impact on the power dissipated in the overhead wires.

The unit of electrical power allows us to roll out a classic physics joke [7] as it's the *Watt*. If a p.d. of 1 V drives a current of 1 A, power produced is 1 W, i.e., 1 W = 1 A · 1 V

Example 11.11 An electric toaster is labelled '220 V 900 W'. When it is operating normally, find (a) the current through it, (b) the resistance in the toaster.

7 Teacher: What is the unit of power?
Student: Watt!
Teacher: Yes, What is the unit of power?

current: $I = \frac{P}{V} = \frac{900}{220} \quad \Rightarrow \quad I \approx 4.09\ \text{A}$
resistance: $R = \frac{V}{I} = \frac{220}{4.09}$, or alternatively, $R = \frac{V^2}{P} = \frac{220^2}{900} \quad \Rightarrow \quad R \approx 53.8\ \Omega$

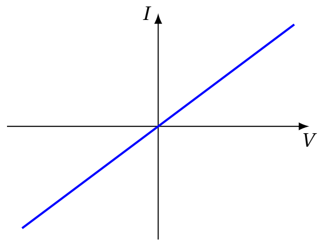
Example 11.12 Two conducting cylinders X and Y are of the same diameter and the same material. The length of the cylinder X is twice of that of the cylinder Y. If the two cylinders are connected in series to a power supply so that the current through them are equal, find the ratio of the electrical powers dissipated in the two cylinders $\frac{P_X}{P_Y}$.

for either cylinder, electrical power: $P = I^2R = I^2 \cdot \frac{\rho L}{A} = I^2 \cdot \frac{4\rho L}{\pi d^2}$
same current I, same diameter d, same resistivity ρ, so power:
$P \propto L \quad \Rightarrow \quad \frac{P_X}{P_Y} = \frac{L_X}{L_Y} = 2$

Electrical components & I-V characteristics

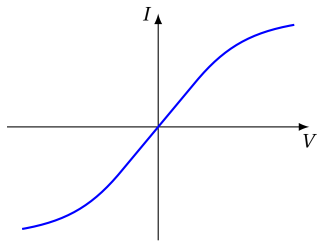
Electrical circuits are all about the interplay of Voltage and Current.
As such the key to understanding how different components behave differently when a p.d. is applied is through an I-V characteristics graph.

Ohmic conductors



The resistance of an ohmic conductor is constant for all currents, so current is directly proportional to p.d.. Metal wires are usually considered as perfect ohmic conductors as the resistance of metal wires is nearly constant at fixed temperature. In a V-I graph, the resistance of an ohmic conductor equals the inverse of the gradient.

Filament lamps

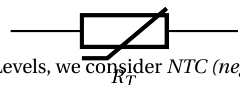


Lamp filaments are usually made from tungsten because it has a very even emission across the visible range when how. It's also capable of sustained high temperatures (a fillament is essentially a demonstration of black-body radiation, which is why some bulbd tend to look yellow - they're not hot enough to be white). For small currents, the filament doesn't get hot enough to emitt light and behaves like an ohmic component. Larger currents cause the filaments to be heated to high temperature and the resistance to increase.[8] To find resistance of lamp filament, we read off values of I and V, then the ratio $\frac{V}{I}$ gives us the resistance.

[8] This is because the vibration of the metal lattice increases as the temperature goes up, free electrons are more likely to be scattered, so the metal becomes less conductive and the current does not increase as much for the same p.d. increase.

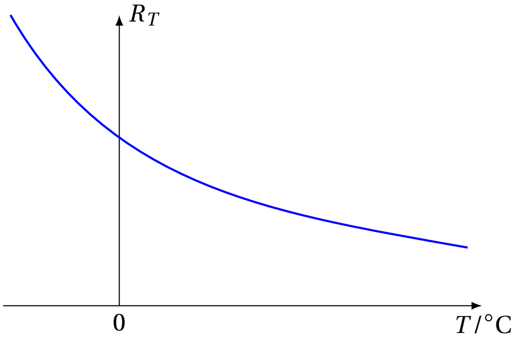
Thermistors

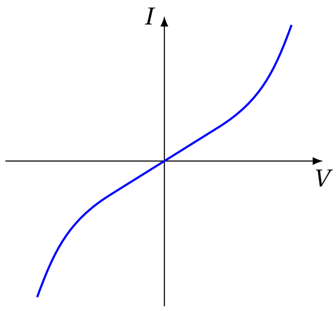
A **Thermistor** has a resistance that varies with temperature in a way you might not expect. As temperature rises, the resistance of the thermistor becomes lower[9]



[9] In A-Levels, we consider *NTC (negative temperature coefficient)* thermistors only. This means their resistance decreases as temperature rises. *PTC (positive temperature coefficient)* thermistors also exist but let's pretend they don't.

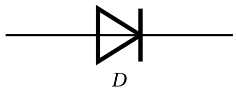
The behaviour of thermistors can be explained in terms of band theory of solids. The material is behaving just like a wire, but there's an additional dominant effect: To put it in simple words, we can say that electrons are bound to atoms and cannot move freely at low temperatures, but they can gain energy and break free from atoms if temperature goes up. There are more free electrons available to conduct electricity, so resistance decreases. At greater currents, current increases more for same p.d. increase.



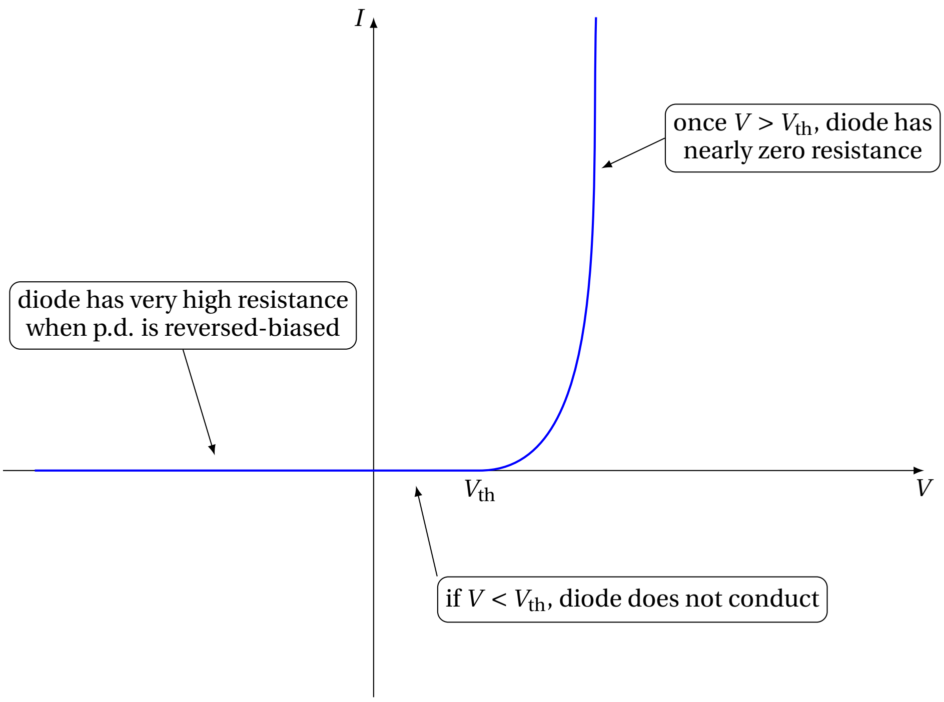


To find resistance of thermistor, we also compute the ratio $\frac{V}{I}$

Diodes



A **diode** only allows current to pass in one direction. When an *ideal* diode is forward-biased, $R_D \to 0$.
When an *ideal* diode is reversed-biased, $R_D \to \infty$.



A practical semiconductor diode become conductive once p.d. reaches a set value V_{th} called the diode's *threshold voltage* (about $0.6 \sim 0.7$ V for a silicon diode). When a p.d. applied is forward-biased and greater

than V_{th}, the diode has very low resistance, so there is a sharp increase in current once $V > V_{\text{th}}$
Though diodes have very high resistance if reversed-biased, if the reverse voltage reaches the *breakdown voltage*, diode would become conductive[10]

[10] Typical breakdown voltage for a diode range from a few volts to several hundred volts, depending on its desired application.

End-of-chapter questions

Question 11.1 A technician reports that the charge of an ion in a solution is measured to be -5.5×10^{-19} C. State and explain whether this measurement is reliable.

Question 11.2 Calculate the current that gives a charge flow of 300 C in one minute.

Question 11.3 The manufacturer of a car battery claims that the battery can supply a steady current of 40 A for one hour when fully-charged. (a) What is the charge flow through a point in the circuit in this time? (b) If a car requires a current of 90 A to start, for how long could the battery be used before recharging?

Question 11.4 A beam of 2.0×10^{10} electrons travel through a distance of 0.50 m and are stopped on an electrode in a time period of 5.0×10^{-8} s. (a) Find the charge collected on the electrode. (b) Find the average current produced by the electron beam.

Question 11.5 State the SI base units of number density n and electric charge q, and hence show that the equation $I = nAqv$ has consistent units on both sides.

Question 11.6 The number density of free electrons in copper is about 8.0×10^{28} m^{-3}. If the current in a copper wire of diameter 0.50 mm is 0.80 A, calculate the average drift speed of electrons in the wire.

Question 11.7 Two coppers wires A and B are joined together and carry a current. If wire A has diameter d and wire B has diameter $2d$, calculate the ratio of the average drift velocity of free electrons in the wires $\frac{v_A}{v_B}$

Question 11.8 From the calculations, we find that the drift speed of electrons in metal wires is around a few mm per second, so it would take a very long time for the free electrons to travel a distance of a few metres in an electric circuit. But when we switch on an electric lamp, it lights up immediately. Give a reason for this observation.

Question 11.9 When a p.d. of 15 V is applied across a wire of resistance 4000 Ω, how many electrons would have passed a point in the wire in one minute?

Question 11.10 A motor has a resistance of 40 Ω and carries a current of 2.0 A. (a) What is the p.d. across? (b) At what rate is electrical energy being dissipated in the motor?

Question 11.11 A cable of 50 m is made of 19 parallel strands of copper wires. Each strand has a diameter of 0.25 mm. Given that the resistivity of copper is 1.7×10^{-8} Ω m, find the resistance of the cable.

Question 11.12 A pencil is used to draw a line of length 20 cm, width 1.5 mm, and thickness 0.10 μm. The resistance of the line is 30 kΩ. What is the resistivity of the material in the pencil?

Question 11.13 One resistor P has a resistance of 16 Ω. A second resistor Q is made of the same material, but has twice the length but half the diameter. What is the resistance of Q?

Question 11.14 A coil is made by wounding a long insulated wire of diameter d onto a cylindrical core of diameter D where $D \gg d$. If the coil has N turns, and the resistivity of the wire is ρ, show that the resistance in the coil is $\frac{4N\rho D}{d^2}$.

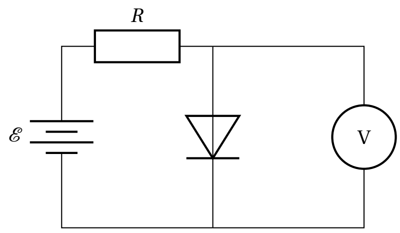
Question 11.15 If we want to determine the resistivity of the material of a length of metal wire, what measurements do we need to take?

Question 11.16 What is the current in a 40 W light bulb when a supply of 200 V is applied?

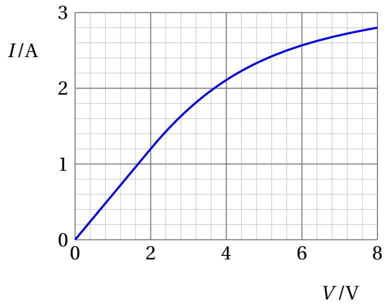
Question 11.17 Fuses of current ratings 1 A, 5 A, 10 A, 20 A are available. Suggest and explain which fuse should be used for a 120 V, 900 W hairdryer.

Question 11.18 A 12 V battery supplies a continuous current of 5.0 A for 3.0 hours. (a) Find the charge supplied by the battery during this time. (b) Find the energy produced from the battery.

Question 11.19 An electric cooker is labelled '240 V, 900 W'. (a) What is the resistance of the cooker's filament at its operating temperature? (b) State and explain whether there is any change in the resistance if it is measured at room temperature.



Question 11.20 The diagram shows an electric circuit incorporating a diode, connected in series with a 5.0 Ω resistor across a battery having an e.m.f. of 3.0 V. If the diode and the voltmeter are both ideal, suggest the reading on the voltmeter (a) for the circuit shown, (b) if the diode is reversed.



Question 11.21 The diagram shows the I-V characteristics of an electrical component P. (a) Suggest whether P a filament lamp or a thermistor. (b) What is the resistance of P for a current of 2.8 A? (c) Draw on the same graph the I-V characteristics for a fixed resistor Q of 4.0 Ω. (d) Compare the power dissipated in P and Q for a p.d. of 4.0 V.

Current Electricity

Kirchhoff's circuit laws

Kirchhoff's laws are two rules that deal with the current and voltage in a circuit
they were first formulated by German physicist *Gustav Robert Kirchhoff* in 1845
the two rules allow for analysis of complex circuits in the field of electrical engineering

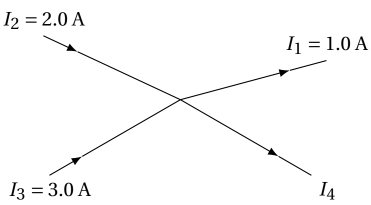
First law

Kirchhoff's first law, also called **Kirchhoff's current law** (KCL), states that for any point in a circuit, sum of currents flowing in equals sum of currents going out:

$$\sum I_{\text{in}} = \sum I_{\text{out}}$$

Recall currents are produced by movement of electric charges so Kirchhoff's first law is therefore a consequence of *conservation of electric charge.*

Example 11.13 The diagram shows part of an electric circuit. The values of I_1, I_2 and I_3 are labelled. Calculate the current I_4.



applying the first law: $I_2 + I_3 = I_1 + I_4$
$\Rightarrow \quad I_4 = 2.0 + 3.0 - 1.0 = 4.0$ A

Second law

Kirchhoff's second law, also known as **Kirchhoff's voltage law** (KVL), states that for any closed loop in a circuit, sum of all e.m.f.'s of supplies is equal to sum of potential differences across resistors:

$$\sum \mathcal{E} = \sum V$$

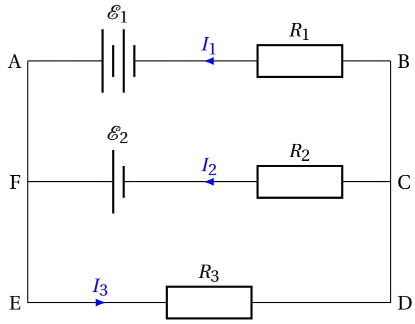
Recall e.m.f. and p.d. are both defined as energy transfer per unit charge. The e.m.f. of a supply is electrical energy produced per unit charge. The p.d. across a resistor gives electrical energy consumed per unit charge. So Kirchhoff's second law is a consequence of *energy conservation.*
Signs for $\mathcal{E}$ and V depend on the choice of loop orientation.
Example 11.14 For the circuit shown, find an expression, in terms of $\mathcal{E}$ and R, for the electric current flowing through the resistors.

apply KVL for the closed loop shown:

$$\mathcal{E}_1 - \mathcal{E}_2 = IR_1 + IR_2$$

$$I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1 + R_2}$$

Example 11.15 The diagram shows a circuit where e.m.f. of the batteries and the values of resistance are all known. Write down the equations for the currents I_1, I_2 and I_3 using Kirchhoff's laws. [11]



apply KCL for point C or F: $I_1 + I_2 = I_3$ ①
apply KVL for loop $AEDBA$: $\mathcal{E}_1 = I_1R_1 + I_3R_3$ ②
apply KVL for loop $FEDCF$: $\mathcal{E}_2 = I_2R_2 + I_3R_3$ ③
in principle, I_1, I_2 and I_3 can be solved from the three equations
one can also apply KVL for loop $AFCBA$ to write: $\mathcal{E}_1 - \mathcal{E}_2 = I_1R_1 - I_2R_2$
this equation looks complex but is simply ②−③, which is not an independent equation
when writing down equations using KVL, choosing simpler loops gives easier equations

[11] The solutions are: $I_1 = \frac{(R_2+R_3)\mathcal{E}_1 - R_3\mathcal{E}_2}{(R_1+R_3)(R_2+R_3)-R_3^2}$, $I_2 = \frac{(R_2+R_3)\mathcal{E}_2 - R_3\mathcal{E}_1}{(R_1+R_3)(R_2+R_3)-R_3^2}$, $I_3 = \frac{R_1\mathcal{E}_2 + R_2\mathcal{E}_1}{(R_1+R_3)(R_2+R_3)-R_3^2}$

Resistor networks

For a network of resistors, it is useful to treat the combination as a single resistor. We can derive formula for series and parallel resistors using Kirchhoff's laws.

Series resistors

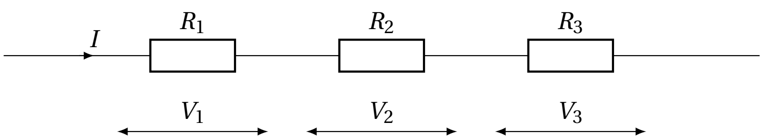


Figure 11.1: three resistors in series

Take three series resistors as shown. The current through each resistor is the same: $I = I_1 = I_2 = I_3$. The e.m.f. however must be split as p.d. across the three resistors: $V_{total} = V_1 + V_2 + V_3$
Dividing both sides by I, we have: $\frac{V_{total}}{I} = \frac{V_1}{I} + \frac{V_2}{I} + \frac{V_3}{I}$
So combined resistance for the three resistors is: $R_{total} = R_1 + R_2 + R_3$
In general, if there are n resistors in series, then:

$$R_{total} = R_1 + R_2 + \cdots + R_n$$

If one of the resistors in a series network increases, then R_{total} would increase. Adding an additional resistor to a network in series, would cause R_{total} to increase.

parallel resistors

let's next take three parallel resistors
same p.d. across each resistor: $V = V_1 = V_2 = V_3$
current is shared: $I_{total} = I_1 + I_2 + I_3$
divide both sides by V: $\frac{I_{total}}{V} = \frac{I_1}{V} + \frac{I_2}{V} + \frac{I_3}{V}$
so combined resistance has: $\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$
in general, for n resistors in parallel:

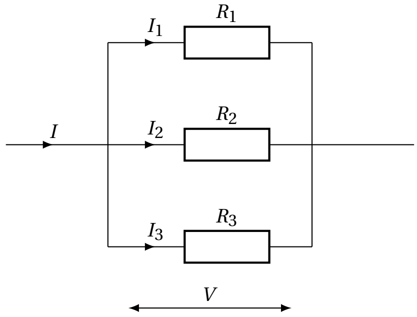
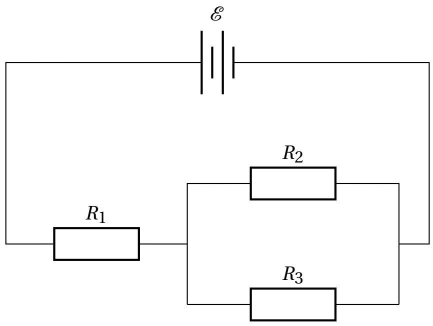


Figure 11.3: three resistors in parallel

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$

if one of the resistors in a parallel network increases, then R_{total} would increase
adding an additional resistor to a network in parallel, then R_{total} would decrease
if n identical resistors R_0 are connected in parallel, then $R_{total} = \frac{1}{n} R_0$



Example 11.16 In the circuit shown, the battery has an e.m.f. $\mathscr{E} = 18$ V, and the resistors have $R_1 = 6.0\ \Omega$, $R_2 = 4.0\ \Omega$ and $R_3 = 12\ \Omega$. (a) Find the total resistance of all external resistors. (b) Find the current through the battery. (c) Find the p.d. across R_1, R_2 and R_3. (d) Find the total power dissipated in R_2 and R_3.

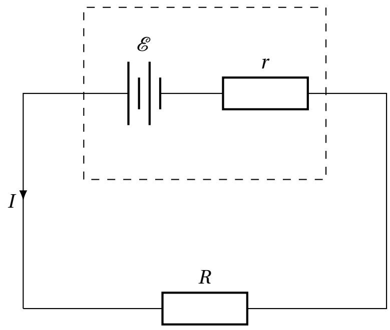
total resistance: $R = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_2}\right)^{-1} = 6.0 + \left(\frac{1}{4.0} + \frac{1}{12}\right)^{-1} = 6.0 + 3.0 = 9.0\ \Omega$

current through battery: $I = \frac{\mathscr{E}}{R} = \frac{18}{9.0} = 2.0$ A
p.d across R_1: $V_1 = IR_1 = 2.0 \times 6.0 = 12$ V
p.d across R_2 and R_3: $V_2 = V_3 = \mathscr{E} - V_1 = 18 - 12 = 6.0$ V
power dissipated in R_2: $P_2 = \frac{V_2^2}{R_2} = \frac{6.0^2}{4.0} = 9.0$ W
power dissipated in R_3: $P_3 = \frac{V_3^2}{R_3} = \frac{6.0^2}{12} = 3.0$ W

practical circuits

power supplies & internal resistance

all real power sources have *internal resistance*
examples are resistance in electrolytic solution of a battery or in coils for a generator
a practical battery can be thought as the combination of an ideal battery with e.m.f. $\mathscr{E}$ and internal resistance r
if this battery is connected to a circuit of external resistance R, applying the Kirchhoff's laws, we write:



$$\mathscr{E} = V_R + V_r = I(R + r)$$

V_R is called the **terminal p.d.** across the battery

V_r is called the **lost volts** in the internal resistance
internal resistance of a battery can be defined as ratio of lost volts to current in battery
when a current flows through a battery, terminal p.d. will be less than battery's e.m.f.
this is because some voltage is lost due to internal resistance
current in the circuit is given by: $I = \frac{\mathscr{E}}{R+r}$

- maximum current when terminals are shorted-out, i.e., when external load $R = 0$
  greatest current is therefore: $I_{\text{max}} = \frac{\mathscr{E}}{r}$
  though this current is limited to some finite value, this still causes great heating effects
  this should be avoided because battery may be destroyed if temperature gets too high
- battery drives no current for an open circuit, i.e., $I \to 0$ if external resistance $R \to \infty$
  in this case, there is no lost volts, so terminal p.d. equals e.m.f.

note that there are several different notions of electrical powers

- total power generated in battery is: $P_{\text{total}} = I\mathscr{E} = I^2(R + r)$
- power delivered to external circuit is: $P_R = IV_r = I^2R$
- power dissipated (rate of thermal energy produced) in battery is: $P_r = IV_r = I^2r$

Example 11.17 A car battery has an e.m.f. of 20 V and an internal resistance of 0.50 Ω. When a starter motor of resistance of 3.50 Ω is connected to the battery, find (a) the current supplied to the motor, (b) the p.d. across the battery terminals, (c) the power at which the motor operates.

> current in circuit: $I = \frac{\mathscr{E}}{R+r} = \frac{20}{3.50+0.50} = 5.0$ A
> terminal p.d.: $V_R = IR = 5.0 \times 3.50 = 17.5$ V
> power of motor: $P_R = IV_R = 5.0 \times 17.5 = 87.5$ W or $P_R = I^2R = 5.0^2 \times 3.50 = 87.5$ W

Example 11.18 A battery of e.m.f. 12 V is connected to a network of total resistance of 22 Ω. The current through the battery is 0.50 A. Find (a) the internal resistance of the battery, (b) the total power produced by battery, (c) the power dissipated in battery.

> lost volts: $V_r = \mathscr{E} - IR = 12 - 0.50 \times 22 = 1.0$ V
> internal resistance: $r = \frac{V_r}{I} = \frac{1.0}{0.50} = 2.0\ \Omega$
> power produced by battery: $P_{\text{total}} = I\mathscr{E} = 0.50 \times 12 = 6.0$ W
> power dissipated in battery: $P_r = IV_r = 0.50 \times 1.0 = 0.50$ W or $P_r = I^2r = 0.50^2 \times 2.0 = 0.50$ W

Example 11.19 An electric heater is operating at a working p.d. of 200 V. A current of 5.0 A is driven by a voltage source of 230 V. The source has an internal resistance of 4.0 Ω and the resistance of the connecting wires is not negligible. Find (a) the lost volts in the source, (b) the resistance of the wires, (c) the useful power from the source, (d) the efficiency of this circuit.

lost volts: $V_r = Ir = 5.0 \times 4.0 = 20$ V
p.d. across wires: $V_{\text{wire}} = \mathscr{E} - V_{\text{heater}} - V_r = 230 - 200 - 20 = 10$ V
resistance of wires: $R_{\text{wire}} = \frac{V_{\text{wire}}}{I} = \frac{10}{5.0} = 2.0\ \Omega$
useful power: $P_{\text{useful}} = P_{\text{heater}} = IV_{\text{heater}} = 5.0 \times 200 = 1000$ W
efficiency: $\eta = \frac{P_{\text{useful}}}{P_{\text{total}}} = \frac{IV_{\text{heater}}}{I\mathscr{E}} = \frac{200}{230} \approx 87\%$

Example 11.20 A cell of e.m.f. $\mathscr{E}$ and internal resistance r is connected in series with a variable resistor R. R is gradually increased from zero. (a) Suggest how the p.d. across the battery terminals change when R is increased. (b) For larger values of R, power delivered to R decreases. Suggest the advantage, despite the low power output, of using this cell in a circuit of larger resistance.

as R increases, current in circuit decreases
less voltage is lost in cell, so terminal p.d. will increase
for same reason, less power is lost in cell, so higher efficiency for the circuit
more explicitly, terminal p.d. $V_R = IR = \frac{\mathscr{E}R}{R+r}$, and efficiency $\eta = \frac{P_R}{P_{\text{total}}} = \frac{I^2R}{I^2(R+r)} = \frac{R}{R+r}$
from these we can tell: if $R \nearrow$, then $\frac{R}{R+r} \nearrow$, so $V_R \nearrow$ and $\eta \nearrow$

power output from a practical battery

power output to external components is

$$P_{\text{out}} = I^2R = \left(\frac{\mathscr{E}}{R+r}\right)^2 R$$

assuming $\mathscr{E}$ and r are constant, then P_{out} only depends on the external load R of the circuit
the diagram below show how P_{out} varies with R

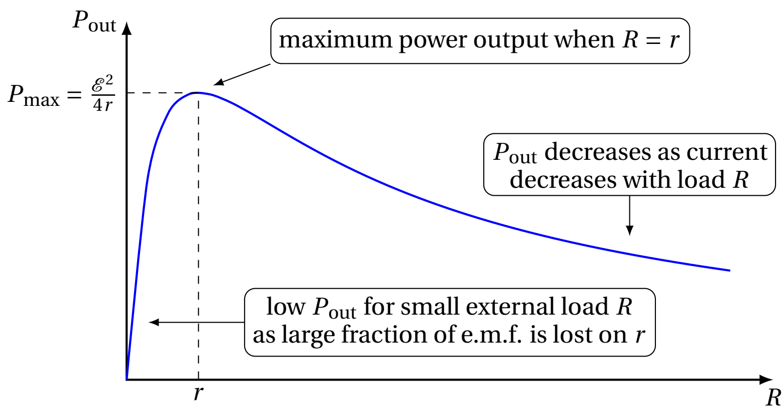


Figure 11.2: variation of output power P_{out} from a power supply to an external resistor R

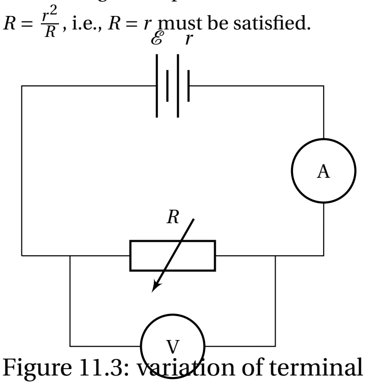
when R is small, only a small fraction of e.m.f. is output as terminal p.d. so most of the power generated is lost on the internal resistance of the source

as R gradually increases, terminal p.d. increases, causing a temporary rise in P_{out}
however, increase in terminal p.d. is compensated by a reduction in electric current
when R continues to increase, current becomes so small that P_{out} gradually tends to zero
maximum power output is archived when $R = r$, this power is given by: $P_{\text{out,max}} = \frac{\mathscr{E}^2}{4r}$ [12]

[12] For any two positive numbers x and y, $x + y \geq 2\sqrt{xy}$ where the equality holds if and only if $x = y$. We then have $P_{\text{out}} = \frac{\mathscr{E}^2 R}{(R+r)^2} = \frac{\mathscr{E}^2}{R+\frac{r^2}{R}+2r} \leq \frac{\mathscr{E}^2}{2\sqrt{R\cdot\frac{r^2}{R}}+2r} = \frac{\mathscr{E}^2}{4r}$. To obtain the greatest power, the condition $R = \frac{r^2}{R}$, i.e., $R = r$ must be satisfied.

measurement of e.m.f. and internal resistance of a power supply

the circuit for determining e.m.f. and internal resistance of an unknown power source is shown on the right
terminal p.d. V_R is measured by the voltmeter
current I in circuit is measured by the ammeter
one can vary R to obtain a set of measurements for I and V
values of V can then be plotted against values of I
since $V_R = \mathscr{E} - Ir$, data points should fall in a straight line
- $\mathscr{E}$ is y-intercept of the graph
- r is given by the negative gradient



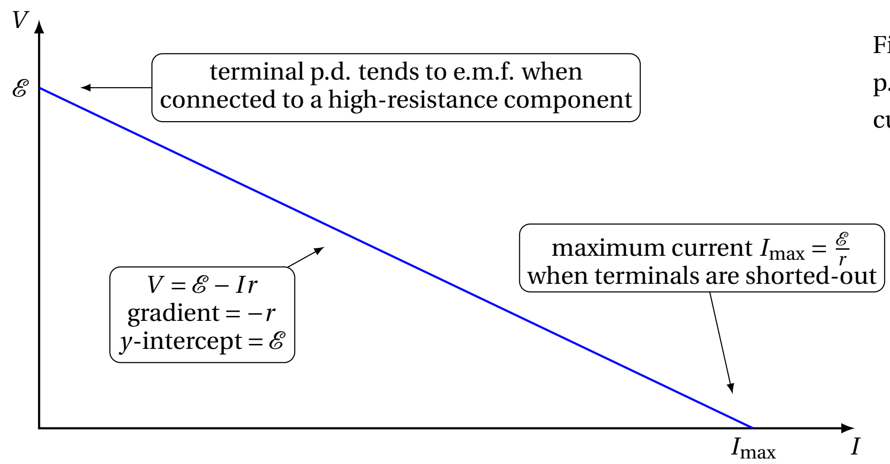
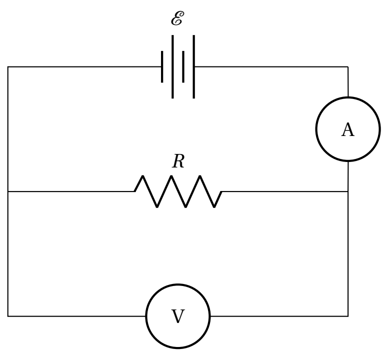


Figure 11.3: variation of terminal p.d across a battery against the current flowing through it

practical ammeters & voltmeters

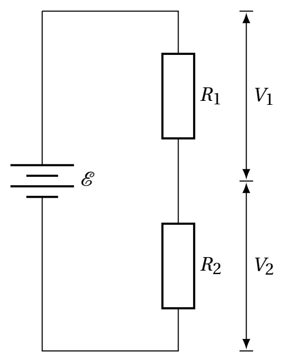
when connected into a circuit, ideal ammeters/voltmeters do not affect original currents
so ideal ammeter has zero resistance, and ideal voltmeter has infinite resistance
but in practice, ammeters have non-zero resistance, voltmeter have finite resistance
to find current through an ammeter/p.d. a voltmeter, treat them as normal resistor
reading on ammeter/voltmeter gives the current through/p.d. across itself



Example 11.21 The diagram shows a simple circuit. The battery has a negligible internal resistance and an e.m.f. of 8.0 V. The resistor has a fixed resistance of 20 Ω. (a) Find the readings shown on the ammeter and the voltmeter if both meters are ideal. (b) Instead, the ammeter has a non-zero resistance of 1.0 Ω and the voltmeter has a resistance of 60 Ω, what are the true readings displayed on the two meters?

if both meters are ideal, then $I = \frac{\mathscr{E}}{R} = \frac{8.0}{20} = 0.40$ A, and $V = \mathscr{E} = 8.0$ V
for non-ideal case, total resistance in circuit: $R_{\text{total}} = R_A + \left(\frac{1}{R} + \frac{1}{R_V}\right)^{-1} = 1.0 + \left(\frac{1}{20} + \frac{1}{60}\right)^{-1} = 16\ \Omega$
current through ammeter: $I = \frac{\mathscr{E}}{R_{\text{total}}} = \frac{8.0}{16} = 0.50$ A
p.d. across voltmeter: $V = \mathscr{E} - V_A = \mathscr{E} - IR_A = 8.0 - 0.50 \times 1.0 = 7.5$ V

potential dividers



one type of useful circuit is the potential divider
a **potential divider** can produce an output voltage that is a fraction of input voltage
a typical potential divider circuit is shown
p.d. across R_1 and R_2 satisfy the relationship:

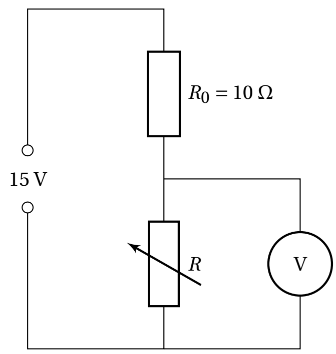
$$\frac{V_1}{V_2} = \frac{IR_1}{IR_2} \quad \Rightarrow \quad \frac{V_1}{V_2} = \frac{R_1}{R_2}$$

we also have the relation: $V_1 + V_2 = \mathscr{E}$
these together give the *potential divider equation*:

$$V_1 = \frac{R_1}{R_1 + R_2} \times \mathscr{E}$$

$$V_2 = \frac{R_2}{R_1 + R_2} \times \mathscr{E}$$

this means R_1 and R_2 divide up the p.d. supplied to them
proportion of p.d. share depends on relative resistance values



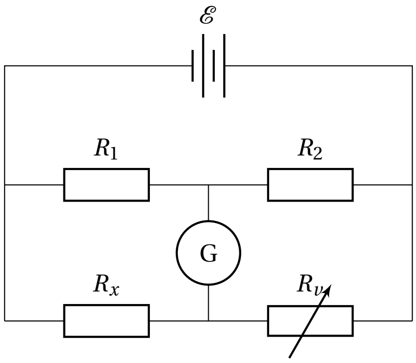
Example 11.22 The diagram shows an electric circuit incorporating a power supply of negligible internal resistance and a high-resistance voltmeter. Determine the range of voltage reading on the voltmeter as the variable resistor R is adjusted over its full range from 0 Ω to 50 Ω.

$V_{\text{min}} = \frac{R_{\text{min}}}{R_{\text{min}} + R_0} \times V_{\text{in}} = \frac{0}{0+10} \times 15 \quad \Rightarrow \quad V_{\text{min}} = 0$ V
$V_{\text{max}} = \frac{R_{\text{max}}}{R_{\text{max}} + R_0} \times V_{\text{in}} = \frac{50}{50+10} \times 15 \quad \Rightarrow \quad V_{\text{max}} = 12.5$ V
so voltage reading ranges from 0 to 12.5 V

bridge circuits

bridge circuits can be designed by altering the potential divider circuit [13]
by balancing two legs of a bridge circuit, an unknown resistance can be measured

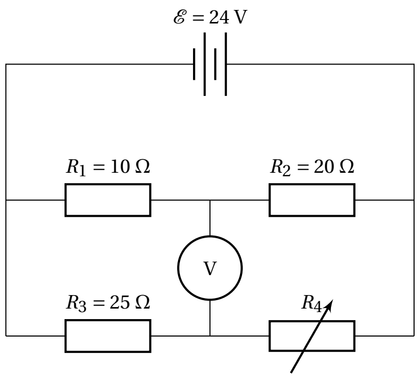
[13] The bridge circuit we will be looking at is known as the *Wheatstone bridge*. The circuit design was invented by British scientist *Samuel Hunter Christie* in 1833 and later improved by another British scientist *Charles Wheatstone* in 1843.



the circuit diagram shows a typical bridge circuit
R_x is the unknown resistance to be measured
resistance of R_1, R_2 and variable resistor R are known
R is adjusted until no current flows through galvanometer, the bridge is then said to be *balanced*
p.d. share between R_1 and R_2 is same as p.d. share between R_x and R:
$\frac{R_1}{R_2} = \frac{R_x}{R}$
so resistance of R_x can be determined to great accuracy [14]

[14] Measurement with the Wheatstone bridge circuit can be extremely accurate because the scheme illustrates the concept of a difference measurement, which can be done to very high accuracy.

Example 11.23 Four resistors are connected to a battery as shown. (a) If the variable resistor R_4 is adjusted to have a resistance of 35 Ω, what is the reading on the voltmeter? (b) If the voltmeter reads zero, what is the resistance for R_4?



(a) p.d. across R_1: $V_1 = \frac{10}{20+10} \times 24 = 8.0\,\text{V}$

p.d. across R_3: $V_3 = \frac{25}{35+25} \times 24 = 10.0\,\text{V}$

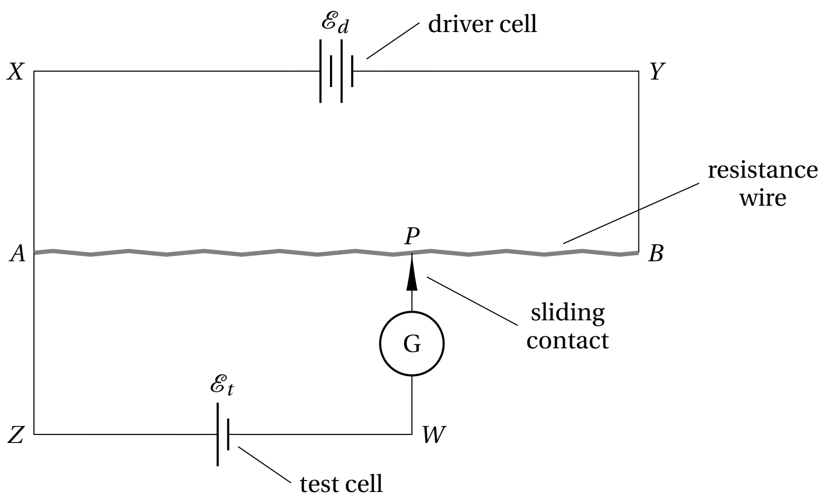
voltmeter reading: $V = V_3 - V_1 = 10.0 - 8.0 = 2.0\,\text{V}$

(b) $V = 0$ means p.d. share between R_1 and R_2 is same as p.d. share between R_3 and R_4

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \quad \Rightarrow \quad \frac{10}{20} = \frac{25}{R_4} \quad \Rightarrow \quad R_4 = 50\,\Omega$$

potentiometer

another useful type of potential divider is the *potentiometer*
p.d.'s or e.m.f.'s can be compared in terms of length quantities with a potentiometer
the diagram shows a potentiometer being used to measure the e.m.f. of a unknown cell
suppose driver cell has no internal resistance and an e.m.f. of $\mathscr{E}_d$ that is already known



to find e.m.f. of test cell, we adjust position of contact P until galvanometer shows zero

taking loop $XAPBYX$, we have: $\mathscr{E}_d = V_{AB} = IR_{AB}$

taking loop $ZAPWZ$, we have: $\mathscr{E}_t = V_{r,t} + V_{AP} = \cancelto{0}{ir_t} + IR_{AP} = IR_{AP}$

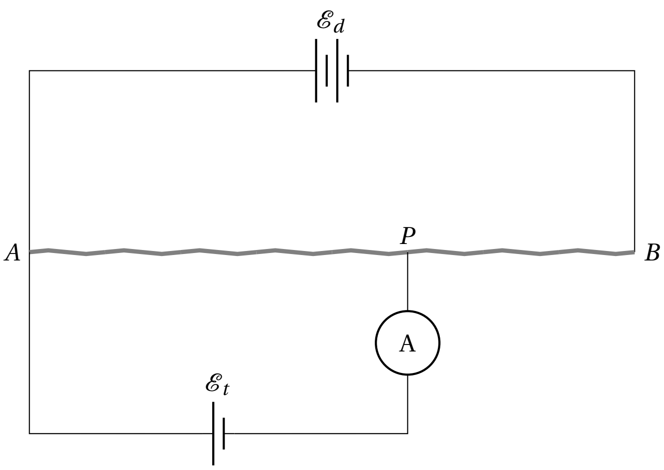
compare the two equations, we find: $\frac{\mathscr{E}_t}{\mathscr{E}_d} = \frac{R_{AP}}{R_{AB}}$

recall resistance of uniform wire is proportional to its length: $\frac{R_{AP}}{R_{AB}} = \frac{AP}{AB}$

now ratio of e.m.f.'s of the two cells are related to ratio of two lengths AP and AB

e.m.f. of test cell is given by: $\mathscr{E}_t = \frac{AP}{AB} \times \mathscr{E}_d$

Example 11.24 In the circuit below, the driver cell has an e.m.f. of 17 V and an internal resistance of 5.0 Ω. AB is a resistance wire of total resistance 80 Ω and length 80 cm. The e.m.f. of a test cell with an unknown internal resistance is to be determined. The moving contact P is adjusted to a position where $AP = 50$ cm such that the ammeter shows no reading. (a) Find the current flowing in the resistance wire. (b) Find the p.d across A and P. (c) State the e.m.f. of the test cell.



current in wire: $I = \frac{\mathscr{E}}{R_{AB}+r_d} = \frac{17}{80+5.0} = 0.20\,\text{A}$

resistance of AP: $R_{AP} = \frac{AP}{AB} \times R_{AB} = \frac{50}{80} \times 80 = 50\,\Omega$

p.d. across AP: $V_{\text{AP}} = IR_{\text{AP}} = 0.20 \times 50 = 10\,\text{V}$

note that there is no need to worry about internal resistance of the test cell

since no current flows through test cell at point of balance, so no lost volts

hence, e.m.f. of test cell: $\mathscr{E}_t = \cancelto{0}{V_{r,t}} + V_{AP} = 10\,\text{V}$

end-of-chapter questions

Kirchhoff's laws

series & parallel resistors

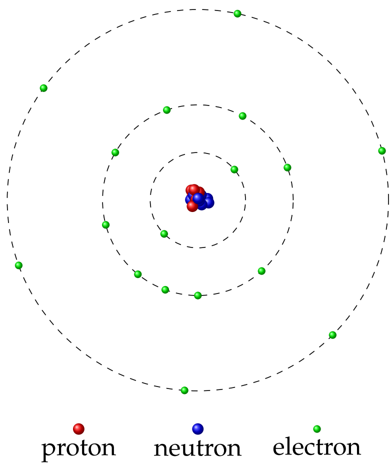
Question 11.22 Use the idea of resistors in series or in parallel to explain (a) why a long wire has more resistance than a short wire, and (b) why a thick wire has less resistance than a thin wire.

12 Sub-atomic Particles

Atomic structure

Atomic model

All matter is composed of tiny particles called **atoms**, each different element has its own type of atom & the diagram below illustrates a simple atomic model.



At the centre of each atom is the **nucleus**. The radius of an atom ∼ $10^{-10} \sim 10^{-9}$ m whereas the radius of a nucleus ∼ $10^{-15} \sim 10^{-14}$ m.
Particles that make up the nucleus are called **nucleons** and they come in two types, **protons** and **neutrons**. Outside the nucleus are the **electrons**, which move around the nucleus in a *cloud*.
Protons, neutrons and electrons are the building blocks of all atoms and hence the building blocks for all matter
The properties of protons, neutrons and electrons are listed below:

- charge of each proton and electron is the elementary charge unit: $e = 1.60 \times 10^{-19}$ C
- neutron has zero charge
- a proton has very similar mass as a neutron
- electron has much smaller mass than a nucleon (proton or neutron)

The α-particle scattering experiment

Hans Geiger, working in Rutherford's lab, did a series experiments in 1908 showing that alpha particles are "scattered" as they pass through thin lay-

subatomic particle	charge	mass	location found
proton	$+e$	$m_p = 1.67 \times 10^{-27}$ kg	nucleus
neutron	0	$m_n = 1.67 \times 10^{-27}$ kg	nucleus
electron	$-e$	$m_e = 9.11 \times 10^{-31}$ kg	in outer atom

ers of mica, and foils of gold and aluminium. In following year, joined by undergraduate Ernest Marsden, they did a series of experiments to untangle confusing results they observed. At the time of the experiment, the atom was thought to be analogous to a plum pudding (as proposed by J. J. Thomson), with the negatively-charged electrons (the plums) studded throughout a positive spherical matrix (the pudding). The scattering in this model was proposed to occur by many repeated collisions with the electrons and the alpha particles should only be deflected by small angles as they pass through.

The plum-pudding model predicts that deflection of α-particles should be uniformly distributed around the central beam, but there is no reason that an α-particle would undergo a large change in its direction of motion, like a cannonball hitting a piece of paper and rebounding backwards. The results of the scattering experiment meant that the plum-pudding model had to be overturned. In order to explain the experimental observations, Rutherford then put forward the new model which now bears his name.

One of the most important experiments they carried out was the **α-particle scattering experiment** in which *Hans Geiger* and *Ernest Marsden* studied atomic structure by firing α-particle beam towards a thin gold foil at the University of Manchester.

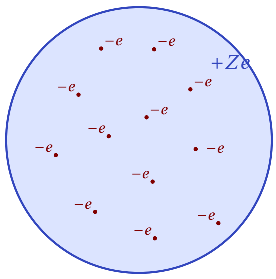


Figure 12.1: The Plum pudding model of the atom.

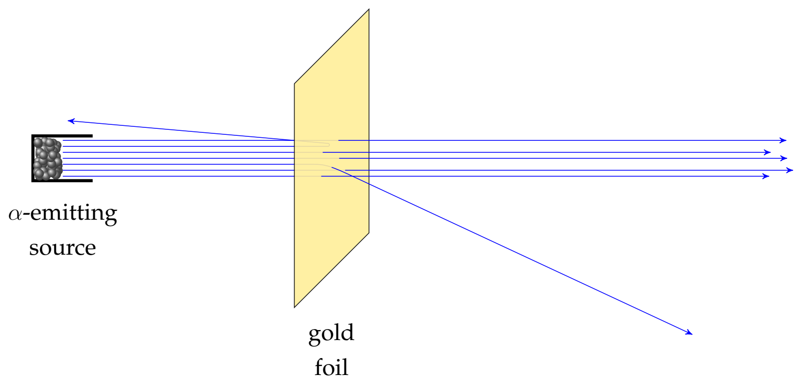


Figure 12.1: Rutherford's α-particle scattering experiment

The main experimental results of the experiments are:

– most α-particles pass straight through the foil with almost no deflection

– very few α-particles (about 1 in 10^4) are deflected by large angles.

To explain the observations, Rutherford concluded the following about nature of atoms

– most space in an atom is empty
(so most α-particles merely pass through empty space without deflection)

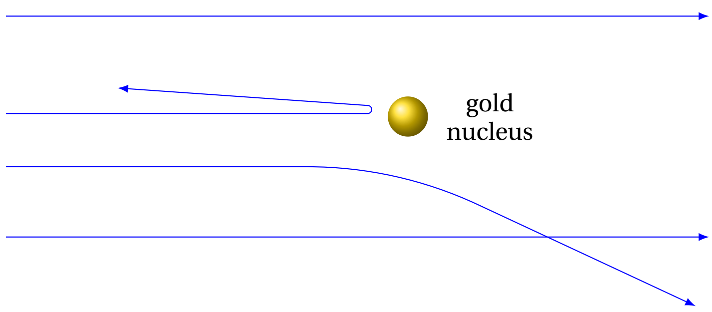


Figure 12.2: paths of α-particles as they approach and pass by a gold nucleus

– there is a tiny positively-charged core, called the nucleus, at centre of an atom
  (so only a small fraction of α-particles would interact with the nucleus)
– almost all mass of the atom is concentrated in the nucleus
  (so interaction is influential if nucleus happens to sit on the path of α-particle beam)

Based on these understandings, Rutherford proposed the atomic model introduced in §12. This model is now known as *Rutherford's model*, or *planetary model of the atom*[1]

Nuclide notation

Nuclides are specific types of atoms or nuclei. A nuclide is uniquely defined by its **proton number** Z and its **nucleon number** A. A nuclide is usually denoted as $^A_Z X$, where X is the chemical symbol for the element.

By definition, Z gives number of protons, A gives number of nucleons, as a consequence, number of neutrons is given by $N = A - Z$. Proton number Z can be further interpreted as **charge number** of the particle . Recall that each proton carries charge $+e$, and neutrons are chargeless so Z further determines electrical charge of the nucleus:

$$Q = +Ze$$

.

This extension will be important when we deal with other particles later in this chapter.

The nucleon number A can be interpreted as **mass number** of the particle , each proton and neutron is of approximately the same mass so the mass of nucleus is determined by total number of protons and neutrons:

$$m = A\mathrm{u}$$

.

where $1\ \mathrm{u} = 1.66 \times 10^{-27}$ kg is the **unified atomic mass unit** [2]

Using this notation, protons, neutrons and electrons can be represented by:

$$^1_1\mathrm{p} \qquad ^1_0\mathrm{n} \qquad ^0_{-1}\mathrm{e}$$

[1] ⚠Rutherford originally suggested that electrons rotate around the nucleus in *circular* orbits due to attractive electrostatic force, like the planets orbit around the sun. This is why Rutherford's model of atomic structure is also called *the planetary model of the atom*.
However, later studies showed that Rutherford's model had its own problems. Classical electromagnetic theory suggests that any charged particle in accelerated motion would radiate energy, so the orbiting electrons will gradually lose all its energy and fall into the nucleus. An atom can never be stable if the Rutherford's model is correct. The full description of the atomic structure requires the understanding of the *quantum* behaviour of electrons, which are described by *probability waves*. Electrons actually do not move in circular orbits but have a certain probability to be anywhere in space at any time, forming an *electron cloud*.

[2] Note that the unified atomic mass (1.661×10^{-27} kg), formally defined as one twelfth of the mass of an carbon-12 atom, is slightly less than the mass of a *free* proton ($m_p = 1.673 \times 10^{-27}$ kg or that of a *free* neutron ($m_n = 1.675 \times 10^{-27}$ kg). This is because energy goes out when protons and neutrons combine to form a nucleus, and this would be associated with a decrease in the mass as suggested by *Albert Einstein's mass-energy equivalence principle*. You can think of u as the average mass of a proton or a neutron when a group of them squeeze together. If you want to find out the mass of a nucleus, you should use u. But if you are dealing with a single proton or a single neutron, you should use m_p and m_n instead.

Example 12.1 The nuclide uranium-238 can be denoted by $^{238}_{92}\mathrm{U}$. (a) State its proton number and its neutron number. (b) Find the charge and mass of the uranium-238 nucleus.

proton number: $Z = 92$
neutron number: $N = A - Z = 238 - 92 = 146$
nuclear charge: $Q = +Ze = +92 \times 1.60 \times 10^{-19} \approx 1.47 \times 10^{-17}$ C
mass of nucleus: $m = Au = 238 \times 1.66 \times 10^{-27} \approx 3.95 \times 10^{-25}$ kg

Example 12.2 Radius of a copper-64 nucleus is 4.8×10^{-15} m. Calculate the mean nuclear density.

$$\rho = \frac{m}{V} = \frac{Au}{\frac{4}{3}\pi r^3} = \frac{64 \times 1.66 \times 10^{-27}}{\frac{4}{3}\pi \times (4.8x10^{-15})^3} \quad \Rightarrow \quad \rho \approx 2.3 \times 10^{17}\ \mathrm{kg\,m^{-3}}$$

this is much greater than density of a copper block (about 8.9×10^3 kg m^{-3})
we can therefore infer that the nucleus indeed takes most mass of the atom

Chemical properties of elements

The chemical properties of an atom depend on the number of electrons, the proton number Z gives us more than just the **atomic number** of the element. In a neutral atom, there are same number of protons as electrons so atoms of same chemical properties, i.e., atoms of same element have same proton number. That said, atoms of same chemical properties can have different nuclei. **Isotopes** of an element have same number of protons but a different number of neutrons.

Example 12.3 State and explain whether carbon-12 ($^{12}_{6}\mathrm{C}$) and carbon-14 ($^{14}_{6}\mathrm{C}$) are different isotopes of the same element.

both carbon-12 and carbon-14 atoms have 6 protons
but carbon-12 has 6 neutrons, carbon-14 has 8 neutrons
same proton number but different neutron number, so different isotopes of same element

Radioactivity

Radioactive decays

If a nucleus is unstable then it may break down by releasing radiation from the nucleus, this process is called **radioactive decay**. The three most common types of radioactive decays are α-decay, β-decay and γ-decay. Radioactive decays are *spontaneous* and decay events are independent of outside conditions, such as temperature, pressure, etc. Whether one

nucleus is about to decay is also independent of the other nuclei around it.
Radioactive decays depend only on the internal stability of the nucleus, they are truly *random*.

Properties of α-, β- & γ-radiation

Nature of α-radiation

α-particles are *helium nuclei*, each made up of two protons and two neutrons
an α-particle therefore carries a positive charge of $2e$, and a mass of 4u
α-particles emitted from a given source travel at almost the same speed
speed of α-particles is usually less than one-tenth of the speed of light

Nature of β-radiation

β-radiation is a beam of high-speed *electrons*
β-particle is negatively charged and very light compared with a nucleus
β-particles travel at very high speeds close to the speed of light

Nature of γ-radiation

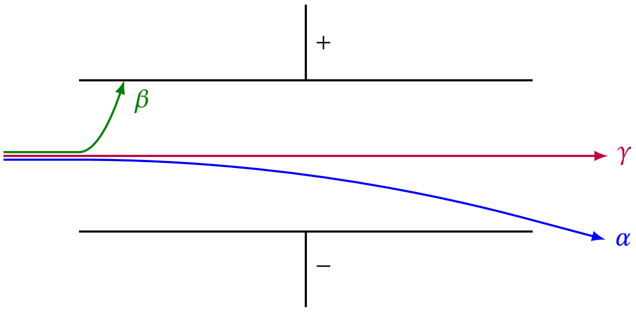
γ-radiation, or γ-photon, is a high-frequency electromagnetic wave
γ-radiation is electrically neutral and has no mass
like any other electromagnetic wave, γ-radiation travels at the speed of light in vacuum

Brief summary for nature of α-, β- & γ-radiation

	α-radiation	β-radiation	γ-radiation
nature	helium nucleus	electron	electromagnetic wave
notation	${}^{4}_{2}\alpha$	${}^{0}_{-1}\beta$	${}^{0}_{0}\gamma$
mass	4u	$\frac{1}{1800}\text{u} \approx 0$	0
charge	$+2e$	$-e$	0
speed	$0.1c$	$0.99c$	c

Example 12.4 If a single beam of α-particles, β-particles and γ-radiation is sent into a uniform electric field as shown, sketch and explain the paths of the beam.

γ-radiation has zero charge, so it is not affected by electric field
α-radiation is positively-charged, so it deflects towards the negative plate
β-radiation is negatively-charged, so it deflects towards the positive plate
β-particles are much lighter, so they have greater deflection



Penetration & ionising power

Different radiation have different abilities to pass through materials, α-, β- and γ- radiation are able to knock out orbital electrons from an atom so radiation can cause the originally neutral atom to become charged. This process is called **ionising**.
Greater ionising ability would imply lower penetration power.
γ-radiation is very penetrative but has very weak ionising power, γ-radiation is not charged, and hence it does not interact strongly with electrons. A γ-photon loses all its energy in single collision and gets absorbed i.e., one γ-photon can only ionise one atom so it has few collisions with electrons, so γ-radiation is not very ionising
α- and β-radiation more ionising than γ-radiation because α- and β-radiation are charged, making them interact with electrons more easily. α-particles are the most highly ionising, recall α-particles are much more massive than electrons so one α-particle can knock out many electrons before it loses all of its kinetic energy. Also α-particles travel at low speeds, so frequent collision events take place in short range. α-radiation therefore has strong ionising ability but weak penetration power.
The table below summarises penetration and ionising power of radioactive radiation:

	α-radiation	β-radiation	γ-radiation
penetration power	low stopped by thick cardboard or a few centimetres of air	fair stopped by metal plates of a few millimetres thick	high stopped by thick lead or concrete of a few centimetres
ionising power	high	fair	low

Laws of conservation

Some of the conserved quantities during any decay process are:

- conservation of *electric charge*
- conservation of *charge number* (Z)
- conservation of *mass-energy*
- conservation of *mass number* (A)

- conservation of *momentum* (review §5.5 if needed)

Decay equations

Using nuclide notation, radioactive decay processes can be described by *decay equations* - the conserved quantities provide a simple guideline to write the decay equations
- sum of charge numbers (Z) on both sides of the equation should add up
- sum of mass numbers (A) on both sides of the equation should add up

The first reported nuclear reaction was credited to Ernest Rutherford. By firing α-particles into pure nitrogen, Rutherford observed the ejection of hydrogen nuclei from the gas, which is now regarded as the discovery of protons. This reaction can be given by:

$$^{14}_{7}\mathrm{N} + ^{4}_{2}\alpha \longrightarrow ^{17}_{8}\mathrm{O} + ^{1}_{1}\mathrm{p}$$

One reaction that is widely used to power nuclear reactors is the induced fission of uranium-235. The reaction can be triggered by bombarding the uranium-235 nuclei with slow neutrons. The uranium nucleus would split up into two lighter nuclei and release a large amount of energy. One of such reactions can be described by:

$$^{235}_{92}\mathrm{U} + ^{1}_{0}\mathrm{n} \longrightarrow ^{141}_{56}\mathrm{Ba} + ^{92}_{36}\mathrm{Kr} + 3^{1}_{0}\mathrm{n}$$

Equation for α-decays

If a nuclide $^{A}_{Z}X$ undergoes α-decays, then we have:

$$^{A}_{Z}X \longrightarrow ^{A-4}_{Z-2}Y + ^{4}_{2}\alpha$$

The original nucleus lost two protons and two neutrons during the decay, on both sides of the equation, there are Z protons and $(A-Z)$ neutrons therefore α-decay simply is an α-particle escaping from the nucleus.

Example 12.5 The process in which a uranium-238 nucleus naturally decays into a thorium-234 nucleus through α-emission can be written as:

$$^{238}_{92}\mathrm{U} \longrightarrow ^{234}_{90}\mathrm{Th} + ^{4}_{2}\alpha$$

Equation for γ-decays

γ-radiation has zero charge and zero mass, so decay equation is very straightforward:

$$^{A}_{Z}X \longrightarrow ^{A}_{Z}X + ^{0}_{0}\gamma$$

since γ-radiation is pure energy, so there is no change of structure of the nucleus in any way

Problems with β-decays

for β-decays, one might attempt to write: ${}^{A}_{Z}X \longrightarrow {}_{Z+1}^{\ \ A}Y + {}^{\ 0}_{-1}\beta$
on the left, nuclide X has Z protons and $(A-Z)$ neutrons
but on the right, nuclide Y has $(Z+1)$ protons and $(A-Z-1)$ neutrons
one neutron has transformed into a proton through giving off an electron: ${}^{1}_{0}\mathrm{n} \longrightarrow {}^{1}_{1}\mathrm{p} + {}^{\ 0}_{-1}\mathrm{e}$???
but how can neutrons and protons transform into one another?!
further experiments show that β-particles emitted for same source have *a range of speeds*
this seems to indicate that energy released from a β-decay can be indefinite
both considerations imply something is wrong, our understanding of β-decay is not complete

- protons and neutrons cannot be the most fundamental particles of nature
  there must exist some more fundamental constituent particles
  these are now known as *quarks*, we will discuss them in the next section
- there must be some other unseen particles released during β-decays
  these particles carry off a fraction of the energy released from the reaction
  for conservation laws to hold, they must be chargeless and very light (maybe massless)
  these ghostly particles are called **neutrinos**, or more precisely, **anti-neutrinos**

we thereby can rewrite the β-decay equation for nuclide ${}^{A}_{Z}X$:

$$ {}^{A}_{Z}X \longrightarrow {}_{Z+1}^{\ \ A}Y + {}^{\ 0}_{-1}\beta + {}^{0}_{0}\bar{\nu} \quad \text{or} \quad {}^{1}_{0}\mathrm{n} \longrightarrow {}^{1}_{1}\mathrm{p} + {}^{\ 0}_{-1}\beta + {}^{0}_{0}\bar{\nu} $$

distribution of kinetic energy of β-particles can be then explained
only total energy of β-particle and anti-neutrino is constant
anti-neutrinos also carry some energy, so β-particles have a range of energies

Example 12.6 Radon ${}^{222}_{86}\mathrm{Rn}$ decays in a sequence of processes to form bismuth ${}^{214}_{83}\mathrm{Bi}$ by emitting α-particles and β-particles. For the decay chain of each radon nucleus, how many α-particles and β-particles are emitted?

let x be number of α-particles emitted and y be number of β-particles emitted

$$ {}^{222}_{86}\mathrm{Rn} \longrightarrow {}^{214}_{83}\mathrm{Bi} + x\cdot{}^{4}_{2}\alpha + y\cdot{}^{\ 0}_{-1}\beta + y\cdot{}^{0}_{0}\bar{\nu} $$

Mass number and charge number are conserved:

$$ \begin{cases} 4x + 214 = 222 \\ 2x - y + 83 = 86 \end{cases} \quad \Rightarrow \quad x = 2,\ y = 1 $$

Fundamental particles

In early 20th century, lots of new subatomic particles were discovered in *cosmic rays* and *particle accelerators*. Many of these particles did not fit into the model where proton, neutrons and electrons are the fundamental building blocks for the physical world. To incorporate these subatomic particles discovered at the time, physicist worked out the **Standard Model of particle physics**.[3]

Particles & anti-particles

For any particle p, there exists an associated **anti-particle** $\bar{p}$
an anti-particle has the same mass but opposite charge to its counterpart
when a particle meets its anti-particle, they **annihilate** each other and produce two γ-photons
combined mass-energy of the pair is converted into electromagnetic energy[4]

Example 12.7 Suggest the properties of the anti-particle of the electron.

> anti-particle of electron has same electron mass: $m_e = 9.11 \times 10^{-31}$ kg
> but it has a positive charge: $q = +e = +1.60 \times 10^{-19}$ C
> anti-particle of electron is usually called a **positron** (positive electron) with the symbol e^+

Quarks & hadrons

One type of the elementary matter particle is the **quark**
quarks come in six varieties, or six **flavours**
they are **up** (u), **down** (d), **strange** (s), **chartm** (c), **top** (t), **bottom** (b)
each quark carries an electric charge, as given in the table below

quarks	u c t	d s b
charge	$+\frac{2}{3}e$	$-\frac{1}{3}e$

note that all quarks carry a *fraction* of the elementary charge unit[5]
Quarks can combine to form composite particles called **hadrons** , there are two ways that several quarks can make up a hadron:

- hadrons can be made up of three quarks (qqq), called **baryons**
  members of the baryon family include protons and neutrons
  a proton consists of two up quarks and one down quark (uud)
  a neutron consists of one up quark and two down quarks (udd)
- hadrons can also be made up of one quark and one anti-quark ($q\bar{q}$), called **mesons**
  you don't need to memorise any example of meson

[3] The theory of the Standard Model was developed in stages since the 1950s, through the work of many great scientists, with the current formulation being finalized in the 1970s.

- *Chen Ning Yang* and *Robert Mills* developed the concept of *gauge theory*, to provide an explanation for the interaction between elementary particles.
- *Sheldon Glashow* proposed the symmetry group that forms the basis of the accepted *electroweak theory*, in which the electromagnetic interaction and weak interactions are unified into a single force.
- *Peter Higgs* and two other groups proposed the *Higgs mechanism* that give rise to mass generation for elementary particles without violating gauge theory through a process called *symmetry breaking*.
- *Steven Weinberg* and *Abdus Salam* incorporated the Higgs mechanism into Glashow's theory, finalizing the unified electroweak theory.
- *David Gross*, *Frank Wilczek*, *David Politze*, and many others developed *quantum chromodynamics*, the theory of the strong interaction, into its modern form.
- …

Being the most accurate accurate scientific theory known to human beings, the Standard Model is now regarded as one of the greatest triumphs of modern physics. The theory not only describes the particles known to scientists, but also predicted new particles, including the *Higgs boson*.

[4] The energy release from the annihilation of particle pairs could be potentially used a energy source. The idea of using antimatter to power spaceships or weapons can be found in many science fiction stories.

[5] Each horizontal line in the table is known as a *generations* of quarks. As you can see, there are three generations of quarks.

you are only required to identify if a particle is a meson given the quark composition

quarks and hadrons are affected by the **strong nuclear force**

strong nuclear force is a very short-ranged attractive force

- it is responsible for hold quarks close together to from hadrons
- it is also responsible for the attraction between hadrons
  for example, protons and neutrons are held together in a nucleus by strong force [6]

quarks only exist in hadrons, i.e., there are no single free quarks in nature[7]

Example 12.8 By reference to the quark composition, explain the electric charge of protons and neutrons.

charge of proton (uud): $q_p = 2q_u + q_d = 2 \times \left(+\frac{2}{3}e\right) + \left(-\frac{1}{3}e\right) = +e$

charge of neutron (udd): $q_n = q_u + 2q_d = \left(+\frac{2}{3}e\right) + 2 \times \left(-\frac{1}{3}e\right) = 0$

Example 12.9 A meson has an electric charge of $+e$ and is known to contain an up quark. Determine a possible flavour of the other quark.

charge of the other quark is: $(+e) - \left(+\frac{2}{3}e\right) = +\frac{1}{3}e$

this could be an anti-down ($\bar{\text{d}}$), an anti-strange ($\bar{\text{s}}$), or an anti-bottom quark ($\bar{\text{b}}$)

Leptons

Another type of the elementary matter particle is the **lepton** - they are not affected by strong nuclear forces. Members of the lepton family include electrons (e), neutrinos (ν), muons (μ), taons (τ)[8]

in A-Level exams, you only need to know electrons (e) and neutrinos (ν)

for each particle, one can assign a **lepton number** L

- a lepton has lepton number $L = +1$
- an anti-lepton has lepton number $L = -1$
- any other particle has lepton number $L = 0$

classification of sub-atomic particles within the A-Level syllabus is summarised below[9]

β-decays

there exists positron, anti-particle of electron, so there are two types of β-particles: β^- and β^+

hence two types of β-decays are possible: β^--decay and β^+-decay

β^--decay revisited

a neutron changes into a proton during a β^--decay: $\boxed{{}^1_0\text{n} \longrightarrow {}^1_1\text{p} + {}^{\;\,0}_{-1}\beta + {}^0_0\bar{\nu}}$

we have learned about the quark structures of protons (uud) and neutrons (udd)

[6] For a small nucleus, strength of strong nuclear force is way greater than the electric repulsion between positively-charged protons, hence the force bind protons and neutrons together forming a stable nucleus. However, since the strong nuclear force has a very short range, the repulsive electrostatic force between the protons might dominate the attractive nuclear force for an over-sized nucleus. Such large nuclei become unstable and are likely to undergo *radioactive decays*.

[7] This is known as the *confinement*, a consequence that is closely related to the fact that interaction between quarks are weak at high energies or smaller length scales but strong at low energies or large length scales, a feature of strong nuclear forces known as *asymptotic freedom*.

[8] Just like the quarks, leptons also come in three *generations*. Each generation of leptons consists of an electron-like particle and its associated neutrino.

[9] This is an over-simplified version of the Standard Model. I only included the particles that you need to know for the A-Level exams. There are a lot of missing pieces that are way beyond the scope of our course, so please don't take this mindmap too seriously.

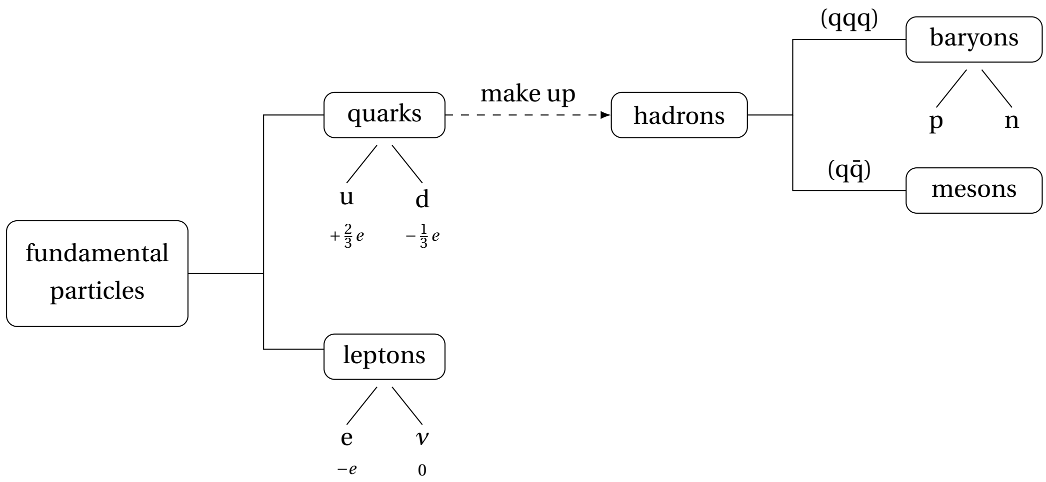


Figure 12.3: a coarse guide for classifying sub-atomic particles in A-Level physics

therefore β^--decay process at a more fundamental level is:

$$d \longrightarrow u + \beta^- + \bar{\nu}$$

β^+-decay

β^+-decay occurs when a proton changes into a neutron and emits a positron

β^+-decay process is described by the equation:

$$^1_1p \longrightarrow ^1_0n + ^0_{+1}\beta + ^0_0\nu$$

in terms of quarks, β^+-decay can be rewritten as:

$$u \longrightarrow d + \beta^+ + \nu$$

nature of β-decay processes is the transformation of quark flavours

the interaction responsible for this transformation is the **weak nuclear force**

there is yet another law of conservation – the **conservation of lepton number**

we can use lepton number to predict whether a neutrino or an anti-neutrino is emitted

Example 12.10 Verify the conservation of (a) charge number, (b) mass number, and (c) lepton number for the β^+-decay process: $^1_1p \longrightarrow ^1_0n + ^0_{+1}\beta + ^0_0\nu$

charge number conservation: $+1 = 0 + (+1) + 0$ ✓
mass number conservation: $1 = 1 + 0 + 0$ ✓
lepton number conservation: $0 = 0 + (-1) + (+1)$ ✓
so this reaction does not violate any of these conservation law

Example 12.11 *Electron capture* is a process where an electron in an atom's inner shell is drawn into the nucleus and combines with a proton. The result is to form a neutron and a neutrino. The process can be given by: ${}^{1}_{1}\mathrm{p} + {}^{0}_{-1}\mathrm{e} \longrightarrow {}^{1}_{0}\mathrm{n} + {}^{0}_{0}\nu$. Show that the process satisfies the conservation of (a) charge number, (b) mass number, and (c) lepton number.

charge number conservation: $+1 + (-1) = 0 + 0$ ✓
mass number conservation: $1 + 0 = 1 + 0$ ✓
lepton number conservation: $0 + 1 = 0 + 1$ ✓
so this reaction does not violate any of these conservation law

End-of-chapter questions

Atomic structure

Question 12.1 List the number of sub-atomic particles in an argon-40 (${}^{40}_{18}\mathrm{Ar}$) atom.

Radioactive decays

Question 12.2 A radioactive source emits a parallel beam of α-, β- and γ-radiation. A detector sensitive to all forms of radiation has been calibrated for background radiation and is placed just 1 cm from the radioactive source. The detector registers 500 counts s^{-1}.
When the detector is 10 cm from the source, the radiation level drops to 200 counts s^{-1}.
When a thin aluminium sheet is placed in front of the detector, the level falls to 50 counts s^{-1}.
In what proportion is the source emitting α-, β- and γ-particles?

Nuclear equations

Question 12.3 An unstable isotope of phosphorus, ${}^{30}_{15}\mathrm{P}$ can be produced by bombarding aluminium, ${}^{27}_{13}\mathrm{Al}$, with α-particles. What is the by-product of this reaction?

Question 12.4 Carbon-14 (${}^{14}_{6}\mathrm{C}$) undergoes β-decay and forms an isotope of nitrogen (N). Write down the decay equation using the nuclide notations.

Question 12.5 Radon-222 (${}^{222}_{86}\mathrm{Rn}$) undergoes a series of decays and forms lead-210 (${}^{210}_{82}\mathrm{Pb}$). How many α-particles and β-particle are emitted during this process?

Fundamental particles

Question 12.6 State the mass and electric charge of an anti-proton.

Question 12.7 π^+ is a particle with quark structure $u\bar{d}$. (a) State the family of this particle. (b) Show that it carries a charge of $+e$. (c) Suggest the charge of its anti-particle.

Question 12.8 Ξ^0 is a particle with quark structure uss. (a) State the family of Ξ^0. (b) Find its electric charge.

Question 12.9 Magnesium-23 ($^{23}_{12}$Mg) can decay into sodium-23 ($^{23}_{11}$Na). (a) Write down the decay equation using nuclide notation, and specify any other product particles produced. (b) Write an equation for this decay in terms of protons and neutrons. (c) Write an equation for this decay in terms of quark composition. (d) State the name of the force that is responsible for this decay.

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