Solids

Name _____ Date _____

By the end of this topic I can...

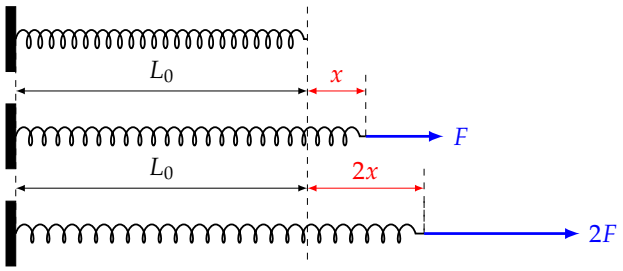
- use Hooke's law, spring constant and elastic potential energy
- combine springs in series and in parallel
- define stress, strain and the Young modulus and use them
- interpret stress–strain graphs for metals, brittle materials, polymers and rubber

Springs

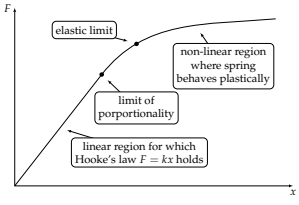
Hooke's law

Hooke's law: the extension of a spring is directly proportional to the load applied, provided the limit of proportionality is not exceeded:

$$F = kx$$



- k is the spring constant (unit $\mathrm{N\,m^{-1}}$); a large k means a stiff spring
- Hooke's law holds for compression as well as extension
- **limit of proportionality**: beyond this point F is no longer proportional to x
- **elastic limit**: beyond this point the spring does not return to its original length; it shows permanent set



Example 7.1 A spring has a natural length of 20.0 cm. When a mass of 250 g is suspended from the spring, the new length of the spring is 26.0 cm. Find the spring constant.

$$k = \frac{F}{x} = \frac{mg}{L - L_0} = \frac{0.250 \times 9.81}{(26.0 - 20.0) \times 10^{-2}} \quad \Rightarrow \quad k \approx 40.9\ \mathrm{N\,m^{-1}}$$

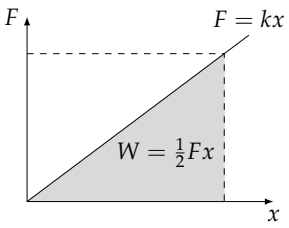
Example 7.2 A spring has a spring constant of 270 N m^{-1}. A mass of 1.2 kg is hung from the spring. When the mass is released from a position where the spring has an extension of 5.0 cm, what is the acceleration of the mass?

$$F_{\text{net}} = F - mg = kx - mg = ma \quad \Rightarrow$$
$$a = \frac{kx - mg}{m} = \frac{270 \times 5.0 \times 10^{-2} - 1.2 \times 9.81}{1.2} \approx 1.44 \text{ m s}^{-2}$$

Elastic potential energy

The force varies as the spring is stretched, so the work done is the area under the F–x graph :

$$E_{\text{p}} = \tfrac{1}{2}Fx = \tfrac{1}{2}kx^2$$



Example 7.3 A steel spring has a spring constant of 20 N cm^{-1}. How much work is needed to stretch it from an extension of 3.0 cm to an extension of 5.0 cm?

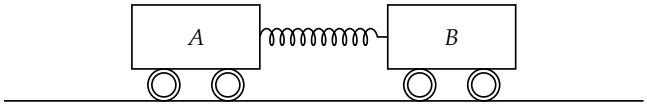
Work done needed equals the increase in elastic potential energy:

$$W = \Delta E_p = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 = \frac{1}{2} \times 2000 \times (0.050^2 - 0.030^2) = 1.6 \text{ J}$$

Example 7.4 A trolley of 400 g can travel freely along a horizontal surface. It is pushed against a spring buffer. Suppose the spring is initially compressed by 5.0 cm under a 20 N force. When the trolley is released, it accelerates until it becomes detached. What is the trolley's final speed?

Elastic potential energy in spring $\Rightarrow$ kinetic energy of trolley

$$\frac{1}{2}Fx = \frac{1}{2}mv^2 \quad \Rightarrow \quad \frac{1}{2} \times 20 \times 0.050 = \frac{1}{2} \times 0.40 \times v^2 \quad \Rightarrow \quad v \approx 1.58 \text{ m s}^{-1}$$



Example 7.5 The same spring (compressed 5.0 cm by a 20 N force) is set between two trolleys A and B of masses 400 g and 600 g. After release, what are their final speeds?

Energy: $\frac{1}{2}Fx = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 \quad \Rightarrow \quad 0.50 = 0.20v_A^2 + 0.30v_B^2$.
Momentum (started at rest): $0.40v_A = 0.60v_B \quad \Rightarrow \quad v_A = 1.5v_B$.
Substituting: $0.20(2.25v_B^2) + 0.30v_B^2 = 0.75v_B^2 = 0.50 \quad \Rightarrow \quad v_B \approx 0.82\ \text{m s}^{-1}$, $v_A \approx 1.2\ \text{m s}^{-1}$.

Combinations of springs

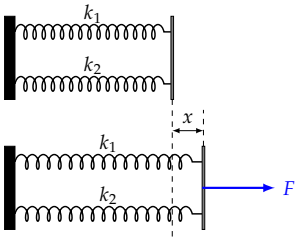
Parallel: same ___extension___, forces add:

$$k = k_1 + k_2 + \cdots$$

Series: same ___force___, extensions add:

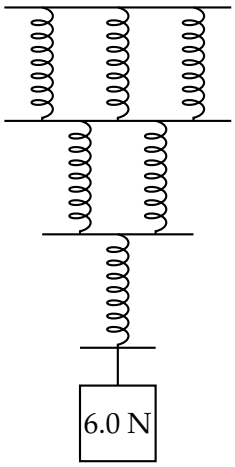
$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$$

- parallel springs are ___stiffer___ than any one spring; series springs are ___less stiff___



Example 7.6 (a) A spring with $k_1 = 20\ \text{N cm}^{-1}$ is connected in series with a second spring with $k_2 = 30\ \text{N cm}^{-1}$. When a force of 60 N is applied, what is the total extension of the combination? (b) The same two springs are now connected in parallel. When a force of 50 N is applied on the combination, what is the extension?

For series connection: $k = \left(\frac{1}{k_1} + \frac{1}{k_1}\right)^{-1} = \left(\frac{1}{20} + \frac{1}{30}\right)^{-1} = 12\ \text{N cm}^{-1} \quad \Rightarrow$ $x = \frac{F}{k} = \frac{60}{12} = 5.0$ cm
or sum up extension of each spring: $x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} = \frac{60}{20} + \frac{60}{30} = 5.0$ cm
for parallel connection: $k = k_1 + k_2 = 20 + 30 = 50\ \text{N cm}^{-1} \quad \Rightarrow \quad x = \frac{F}{k} = \frac{50}{50} = 1.0$ cm □



Example 7.7 A set of identical springs are set up as shown. Each individual spring extends by 1.0 cm under a load of 1.0 N. Assume the limit of proportionality is not exceeded, what is the total extension for this combination when a load of 6.0 N is applied?

For a single spring: $k = 1.0\ \text{N cm}^{-1}$, the combined spring constant: $k_{\text{total}} = \left(\frac{1}{3k} + \frac{1}{2k} + \frac{1}{k}\right)^{-1}$

$$k_{\text{total}} = \left(\frac{1}{3.0} + \frac{1}{2.0} + \frac{1}{1.0}\right)^{-1} = \frac{6}{11} \approx 0.545\ \text{N cm}^{-1}$$

Total extension: $x = \frac{F}{k_{\text{total}}} = \frac{6.0}{0.545} = 11\ \text{cm}$

Alternatively, we can find and add extensions of each layer. In particular, the top layer withstands a force of 6.0 N shared by three springs, so each spring has a force of 2.0 N, extension is 2.0 cm. Similarly, the other two layers extend by 3.0 cm and 6.0 cm respectively, hence, total extension: $x = 2.0 + 3.0 + 6.0 = 11\ \text{cm}$

Stress, strain and the Young modulus

Tensile stress is the force per unit cross-sectional area :

$$\sigma = \frac{F}{A} \qquad [\sigma] = \text{Pa}$$

Tensile strain is the extension per unit original length :

$$\epsilon = \frac{x}{L} \qquad \text{no unit}$$

Example 7.8 A copper wire has a cross-sectional area of $1.5 \times 10^{-6}\ \text{m}^2$. The breaking stress of the wire is 2.0×10^8 Pa. Find the breaking force.

$$\sigma = \frac{F}{A} \quad \Rightarrow \quad F = \sigma A = 2.0 \times 10^8 \times 1.5 \times 10^{-6} = 3.0 \times 10^2\ \text{N}$$

The **Young modulus** is the ratio of stress to strain (below the limit of proportionality):

$$E = \frac{\sigma}{\epsilon} = \frac{FL}{Ax} \qquad [E] = \text{Pa}$$

- the Young modulus is a property of the material, independent of its shape or size

- rearranging gives the spring constant of a wire:

$$k = \frac{EA}{L}$$

Example 7.9 A 200 N tensile force is applied on a steel wire of 1.5 m, the wire extends by 5.0 mm. The diameter of the cross section is 0.60 mm. What is the Young modulus of the steel wire?

$$E = \frac{\text{F/A}}{\text{x/L}} = \frac{FL}{Ax} = \frac{200 \times 1.5}{\pi \times (0.30 \times 10^{-3})^2 \times 5.0 \times 10^{-3}} \quad \Rightarrow \quad E \approx 2.1 \times 10^{11}\ \text{Pa}$$

Example 7.10 A copper wire of length 2.0 m is under a stress of 7.8×10^7 Pa. Given that the Young modulus of copper is 1.2×10^{11} Pa, what is (a) the strain of the wire, (b) the extension of the wire?

strain: $\epsilon = \frac{\sigma}{E} = \frac{7.8 \times 10^7}{1.2 \times 10^{11}} \quad \Rightarrow \quad \epsilon = 6.5 \times 10^{-4} = 0.065\%$

extension: $x = \epsilon L = 6.5 \times 10^{-4} \times 2.0 \quad \Rightarrow \quad x = 1.3 \times 10^{-3}$ m

Example 7.11 Several blocks of steel are used to support a bridge. Each block has a height of 30 cm and a cross section of 15 cm $\times$ 15 cm. The steel block is designed to compress 2.0 mm when the maximum load is applied. Given that the Young modulus of steel is 2.1×10^{11} Pa, what is the maximum load that can be supported by one block?

$$E = \frac{FL}{Ax} \quad \Rightarrow \quad F = \frac{EAx}{L} = \frac{2.1 \times 10^{11} \times (0.15 \times 0.15) \times 2.0 \times 10^{-3}}{0.30}$$

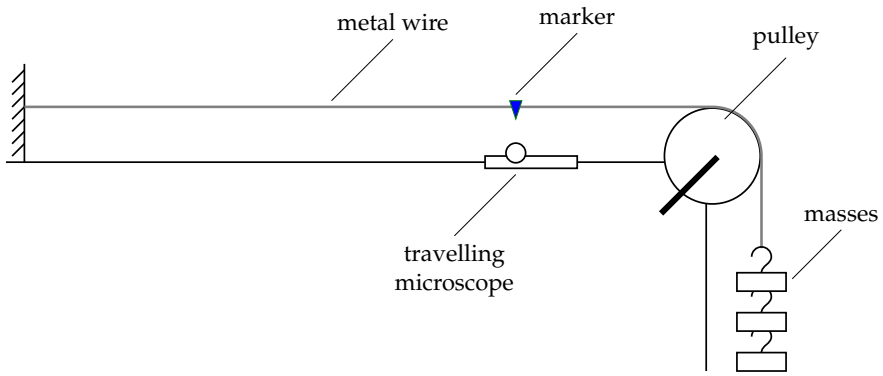
$$\Rightarrow \quad F \approx 3.15 \times 10^7\ \text{N}$$

Example 7.12 Two metal wires A and B are of the same length and they extend by the same amount under the same load. Given that Young modulus of wire A is twice of B, what is the ratio of their diameters?

$$E = \frac{FL}{Ax} \quad \Rightarrow \quad A = \frac{1}{4}\pi d^2 = \frac{FL}{Ex} \quad \Rightarrow$$

$$d^2 \propto \frac{1}{E} \quad \Rightarrow \quad \frac{d_A}{d_B} = \sqrt{\frac{E_B}{E_A}} = \frac{1}{\sqrt{2}}$$

Measuring the Young modulus

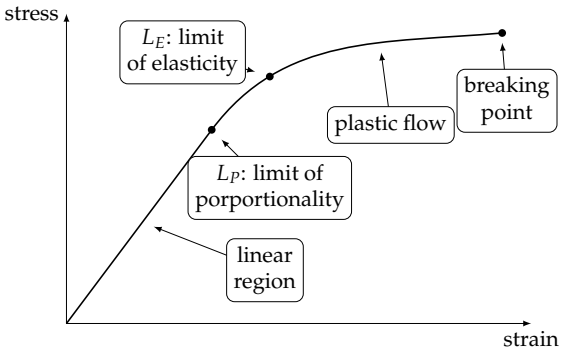


• measure the original length L with a metre rule, the diameter d with a micrometer (several places), so $A = \frac{1}{4}\pi d^2$

• add masses m in steps, giving $F = mg$; measure the extension x at the marker

• plot stress against strain : the gradient of the straight section is the Young modulus

Stress–strain graphs



• up to the limit of proportionality, stress is proportional to strain

• **ductile** materials show plastic deformation before breaking; **brittle** materials break with little or no plastic deformation

• the area under a stress–strain graph is the energy per unit volume (work done per unit volume)

• a wire stretched beyond its elastic limit shows permanent set when unloaded; the area between the loading and unloading curves is the energy lost as heat

Metals and polymers

• **metals:** positive ions in a sea of delocalised electrons; plastic deformation occurs as dislocations move through the lattice

• **work hardening** and **alloying** stop dislocations moving, so the metal has a higher yield stress but is less ductile

• **polymers:** long chains; the chains untangle and straighten (large strain for small stress), then become taut and stiff

• **rubber:** the loading and unloading curves differ (hysteresis); the area between them is the energy dissipated as thermal energy

Practice questions

Question 7.1 A spring of force constant 500 N m^{-1} stretches to a length of 30 cm when a 40 N weight hangs from it. Find its natural length.

$x = \frac{F}{k} = \frac{40}{500} = 0.080$ m $= 8.0$ cm. Natural length $= 30 - 8.0 = 22$ cm.

Question 7.2 Two springs have $k_1 = 8.0$ N cm^{-1} and $k_2 = 12$ N cm^{-1}. Find the total extension under a load of 60 N when they are connected (a) in parallel, (b) in series.

(a) $k = 8.0 + 12 = 20$ N cm^{-1}, so $x = \frac{60}{20} = 3.0$ cm.

(b) $\frac{1}{k} = \frac{1}{8} + \frac{1}{12} \quad \Rightarrow \quad k = 4.8$ N cm^{-1}, so $x = \frac{60}{4.8} = 12.5$ cm.

Question 7.3 A 100 N weight compresses a spring by 2.0 cm. Assuming Hooke's law, how much energy is stored in the spring?

$E_\mathrm{p} = \frac{1}{2}Fx = \frac{1}{2} \times 100 \times 0.020 = 1.0$ J.

Question 7.4 A 160 g trolley is pushed against a spring buffer of force constant 90 N m^{-1}, compressing it by 7.0 cm, then released from rest on a frictionless track. Find (a) the initial acceleration, (b) the final speed.

(a) $a = \dfrac{kx}{m} = \dfrac{90 \times 0.070}{0.160} \approx 39$ m s^{-2}.

(b) $\frac{1}{2}kx^2 = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = x\sqrt{\dfrac{k}{m}} = 0.070\sqrt{\dfrac{90}{0.160}} \approx 1.7$ m s^{-1}.

Question 7.5 A cable of diameter 1.5 mm carries a tension of 200 N. Find the stress in the cable.

$\sigma = \dfrac{F}{A} = \dfrac{200}{\pi(0.75 \times 10^{-3})^2} \approx 1.1 \times 10^8$ Pa.

Question 7.6 A wire of natural length 1.8 m and diameter 0.70 mm hangs from a ceiling. A 6.5 kg mass on its lower end extends it by 2.7 mm. Find (a) the strain, (b) the Young modulus.

(a) $\epsilon = \dfrac{2.7 \times 10^{-3}}{1.8} = 1.5 \times 10^{-3}$.

(b) $\sigma = \dfrac{6.5 \times 9.81}{\pi(0.35 \times 10^{-3})^2} = 1.66 \times 10^8$ Pa, so $E = \dfrac{\sigma}{\epsilon} = \dfrac{1.66 \times 10^8}{1.5 \times 10^{-3}} \approx 1.1 \times 10^{11}$ Pa.

Question 7.7 A wire of diameter 0.50 mm and Young modulus 0.18 TPa is lengthened by 0.20% by a force F. Find (a) the stress, (b) F.

(a) $\sigma = E\epsilon = 0.18 \times 10^{12} \times 0.0020 = 3.6 \times 10^8$ Pa.
(b) $F = \sigma A = 3.6 \times 10^8 \times \pi(0.25 \times 10^{-3})^2 \approx 71$ N.

Question 7.8 Rods A and B of the same diameter are joined end to end and hang vertically. Rod B is twice as long as A but has half the Young modulus. What is the ratio of the extension of A to the extension of B?

The force is the same in both and A is the same, so $x = \dfrac{FL}{AE} \propto \dfrac{L}{E}$.
$\dfrac{x_A}{x_B} = \dfrac{L_A/E_A}{L_B/E_B} = \dfrac{1/1}{2/\frac{1}{2}} = \dfrac{1}{4}$.