Oscillations

Name _____ Date _____

By the end of this topic I can…

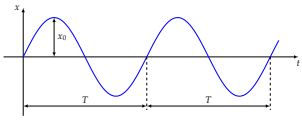
- describe oscillations using amplitude, period, frequency and phase difference
- define simple harmonic motion and use $a = -\omega^2 x$ and $x = x_0 \sin(\omega t + \phi)$
- use v, a and energy relations for SHM, and the periods of a mass–spring system and a pendulum
- describe damping, free and forced oscillations and resonance

Describing oscillations

_____Oscillation_____ is a repetitive back and forth motion about an _____equilibrium position_____.

At the equilibrium position all the forces balance: released from rest there, the object stays at rest. Examples: a pendulum, a vibrating string, a swing.

Amplitude, period, frequency



Displacement–time graph of a typical oscillator.

- **Displacement** x: the distance from the _____equilibrium position_____
- **Amplitude** x_0: the _____maximum displacement_____ from equilibrium
- **Period** T: the time for _____one complete oscillation_____
- **Frequency** f: the number of oscillations _____per unit time_____:

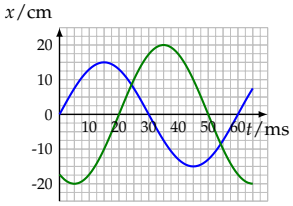
$$f = \frac{1}{T}$$

Phase

The **phase angle** ϕ tells how far through its cycle an oscillator is. Unit: _____radian_____. Think of it as the fraction of a complete cycle, not as a geometric angle.

The **phase difference** compares two oscillators:

$$\Delta\phi = \frac{\Delta t}{T} \times 2\pi$$



Two oscillations (time in ms, displacement in cm).

Example 20.1 Compare the two oscillations in the x–t graph.

Both have period $T = 60$ ms, so frequency $f = \dfrac{1}{60 \times 10^{-3}} \approx 16.7$ Hz.
Different amplitudes: $x_0 = 15$ cm and $x_0 = 20$ cm.
Time difference $\Delta t = 20$ ms, so $\Delta\phi = \dfrac{\Delta t}{T} \times 2\pi = \dfrac{20}{60} \times 2\pi = \dfrac{2\pi}{3}$ rad

Simple harmonic motion

Restoring force

In any oscillation the acceleration always points towards the equilibrium position. The resultant force that does this is the **restoring force**; at equilibrium it is zero.

Simple harmonic motion (SHM): the acceleration is always proportional to the displacement from the equilibrium position, and in the opposite direction to it.

SHM approximates a pendulum, molecular vibrations and more. Complicated motions can be built from a set of simple harmonics.[1]

[1] This is Fourier analysis. For example, uniform circular motion is two SHMs, in the x- and y-directions.

Equation of motion

The defining equation of SHM:

$$a = -\omega^2 x$$

ω is a constant, and the minus sign shows a and x are in opposite directions.

The solution[2] is

$$x = x_0 \sin(\omega t + \phi)$$

where x_0 is the amplitude, ω the angular frequency and ϕ the phase angle.

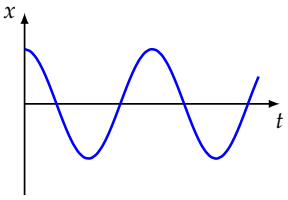
[2] Since $a = \mathrm{d}^2x/\mathrm{d}t^2$, the defining equation is the second-order differential equation $\mathrm{d}^2x/\mathrm{d}t^2 + \omega^2 x = 0$.

- **angular frequency**:

$$\omega = \frac{2\pi}{T} = 2\pi f \qquad [\omega] = \text{rad s}^{-1}$$

- ω depends only on the physical constants of the system; a freely oscillating system has a fixed **natural frequency**

• ϕ depends on the initial conditions; choose sine or cosine so that ϕ is not needed

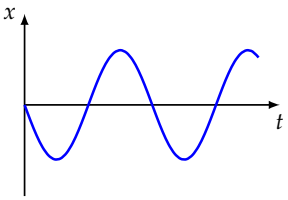


Released from $x = +x_0$: a cosine.

Example 20.2 A simple harmonic oscillator is displaced by 6.0 cm and released at $t = 0$. Its period is 0.80 s. State its displacement–time equation.

$\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.80} = \dfrac{5\pi}{2}$ rad s^{-1}

At $t = 0$, $x = +x_0$, so use a cosine: $x = x_0 \cos \omega t = 6.0 \cos\left(\dfrac{5\pi}{2} t\right)$



At $t = 0$ at equilibrium, moving in the negative direction.

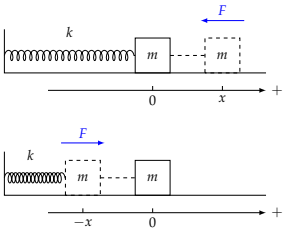
Example 20.3 An oscillator at equilibrium is given a speed in the negative direction at $t = 0$. Its frequency is 1.5 Hz and amplitude 5.0 cm. State its displacement–time equation.

$\omega = 2\pi f = 2\pi \times 1.5 = 3\pi$ rad s^{-1}

At $t = 0$, $x = 0$ and $v < 0$, so use a negative sine: $x = -x_0 \sin \omega t = -5.0 \sin(3\pi t)$

Examples of simple harmonic oscillators

Mass–spring oscillator



Restoring force on a mass–spring oscillator.

A block of mass m on an ideal spring. The spring's restoring force obeys Hooke's law, $F = kx$, directed opposite to the displacement. So

$$F_{\text{net}} = ma \quad \Rightarrow \quad -kx = ma \quad \Rightarrow \quad a = -\frac{k}{m}x$$

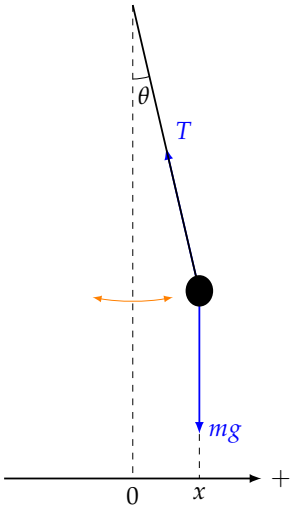
k and m are constant, so $a \propto x$ with a minus sign: it is SHM. Comparing with $a = -\omega^2 x$:

$$\omega = \sqrt{\frac{k}{m}} \quad \text{so} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

The period depends only on m and k, not on the amplitude.

- larger m: longer period (more inertia, slower)
- larger k: shorter period (stiffer spring, larger restoring force)

Simple pendulum



Simple pendulum.

A bob on a light string swings in SHM for small angles. Then the bob has almost no vertical displacement:

vertically: $T\cos\theta \approx mg$, so the tension $\approx mg$

horizontally: $-T\sin\theta = ma$ with $\sin\theta = x/L$, so $a \approx -\dfrac{g}{L}x$

Comparing with $a = -\omega^2 x$ gives $\omega = \sqrt{g/L}$:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

The period depends on L and g only, not on the amplitude (for small angles).

- longer pendulum: longer period
- smaller g: longer period (with $g = 0$ the bob would not move: $T \to \infty$)

Question 20.1 A cylindrical tube of mass m and cross-sectional area A floats upright in a liquid of density ρ. For a small vertical displacement y, the resultant force on it has magnitude $F_{\text{net}} = \rho g A y$. (a) Show that it performs SHM. (b) Find an expression for the frequency.

(a) The resultant force acts to restore the tube, so it is opposite to y: $ma = -\rho g A y \quad\Rightarrow\quad a = -\dfrac{\rho g A}{m}y$. The acceleration is proportional to y and opposite to it: SHM.
(b) $\omega^2 = \dfrac{\rho g A}{m}$, so $f = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{\rho g A}{m}}$

Question 20.2 A glider on a frictionless horizontal air track bounces elastically off buffers at the ends. Does it describe SHM? Explain.

No. Between the buffers it moves at constant velocity, so its acceleration is zero and does not depend on displacement; at the buffers the acceleration is large and sudden. Acceleration is not proportional to displacement.

Velocity, acceleration and energy

Velocity and acceleration

From $x = x_0 \sin(\omega t + \phi)$, velocity is the
rate of change of displacement :

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} \quad \Rightarrow \quad \boxed{v = \omega x_0 \cos(\omega t + \phi)}$$

Eliminating the sine and cosine: $v^2 + \omega^2 x^2 = \omega^2 x_0^2$, so

$$\boxed{v = \pm\omega\sqrt{x_0^2 - x^2}}$$

- at $x = 0$ the speed is a maximum : $\boxed{v_{\max} = \omega x_0}$
- at $x = \pm x_0$ the oscillator is momentarily at rest : $v = 0$

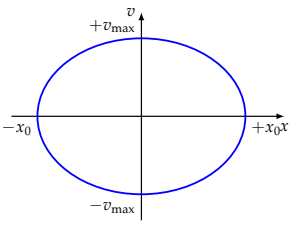
Acceleration is the rate of change of velocity:

$$\boxed{a = -\omega^2 x_0 \sin(\omega t + \phi)}$$

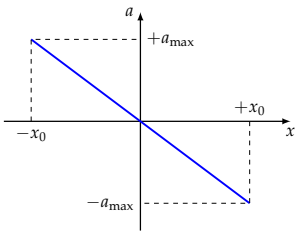
which is just $a = -\omega^2 x$ again.

- at $x = 0$ the acceleration is zero
- at $x = \pm x_0$ it is a maximum : $\boxed{a_{\max} = \omega^2 x_0}$

time t	0	$\frac{1}{4}T$	$\frac{1}{2}T$	$\frac{3}{4}T$	T
$x = x_0 \sin \omega t$	0	+max	0	−max	0
$v = \omega x_0 \cos \omega t$	+max	0	−max	0	+max
$a = -\omega^2 x$	0	−max	0	+max	0



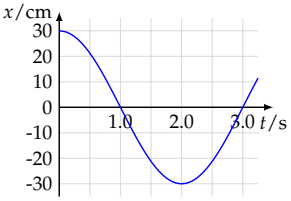
v–x graph: an ellipse.



a–x graph: a straight line with negative gradient.

Example 20.4 A pendulum swings from one end to the other, a distance of 6.0 cm, in 1.0 s. (a) State the period and amplitude. (b) Find the greatest speed. (c) Find the speed when $x = 1.2$ cm.

(a) $T = 2 \times 1.0 = 2.0$ s and $x_0 = \frac{1}{2} \times 6.0 = 3.0$ cm
(b) $\omega = \dfrac{2\pi}{T} = \pi$ rad s^{-1}, $v_{\max} = \omega x_0 = \pi \times 3.0 \approx 9.4$ cm s^{-1}
(c) $v = \omega\sqrt{x_0^2 - x^2} = \pi\sqrt{3.0^2 - 1.2^2} \approx 8.6$ cm s^{-1}



x–t graph (cm, s).

Example 20.5 For the SHM in the graph, find (a) its speed at $t = 0$, (b) its greatest speed, (c) its acceleration at $t = 1.0$ s.

(a) At $t = 0$, $x = +x_0$: gradient zero, so $v = 0$.
(b) $x_0 = 30$ cm, $T = 4.0$ s, $\omega = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$ rad s^{-1}; $v_{\max} = \omega x_0 = \dfrac{\pi}{2} \times 30 \approx 47$ cm s^{-1}
(c) At $t = 1.0$ s, $x = 0$ (equilibrium), so $a = 0$.

Question 20.3 A car-engine piston moves in SHM, completing 3000 oscillations per minute with amplitude 4.0 cm. Find (a) the greatest speed, (b) the greatest acceleration.

$f = \dfrac{3000}{60} = 50$ Hz, so $\omega = 2\pi f \approx 314$ rad s^{-1}
(a) $v_{\max} = \omega x_0 = 314 \times 0.040 \approx 12.6$ m s^{-1}
(b) $a_{\max} = \omega^2 x_0 = 314^2 \times 0.040 \approx 3.9 \times 10^3$ m s^{-2}

Vibrational energy

For an ideal mass–spring oscillator the energy has two parts:

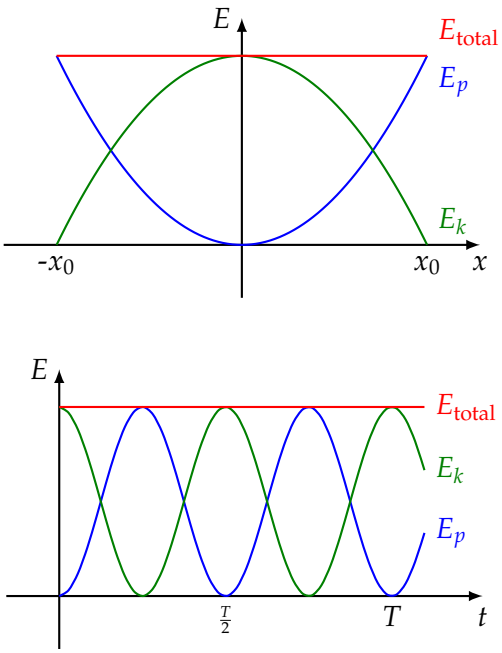
- kinetic: $E_k = \frac{1}{2}mv^2 =$ $\frac{1}{2}m\omega^2(x_0^2 - x^2)$
- elastic potential: $E_p = \frac{1}{2}kx^2 =$ $\frac{1}{2}m\omega^2 x^2$

Adding them:

$$E = \tfrac{1}{2}m\omega^2 x_0^2$$

This holds for __all__ simple harmonic oscillators, and the total is __constant__ for an ideal system.

- at $x = 0$: E_k is __maximum__, $E_p = 0$ (all kinetic)
- at $x = \pm x_0$: $E_k = 0$, E_p is __maximum__ (all potential)



Example 20.6 A 150 g block on a spring oscillates with period 0.80 s and maximum displacement 12 cm. Find the vibrational energy.

$$E = \tfrac{1}{2}m\omega^2 x_0^2 = \tfrac{1}{2} \times 0.15 \times \frac{4\pi^2}{0.80^2} \times 0.12^2 \approx 6.7 \times 10^{-2}\ \text{J}$$

Question 20.4 An oscillator given 20 mJ reaches an amplitude of 8.0 cm. What energy is needed to double the amplitude?

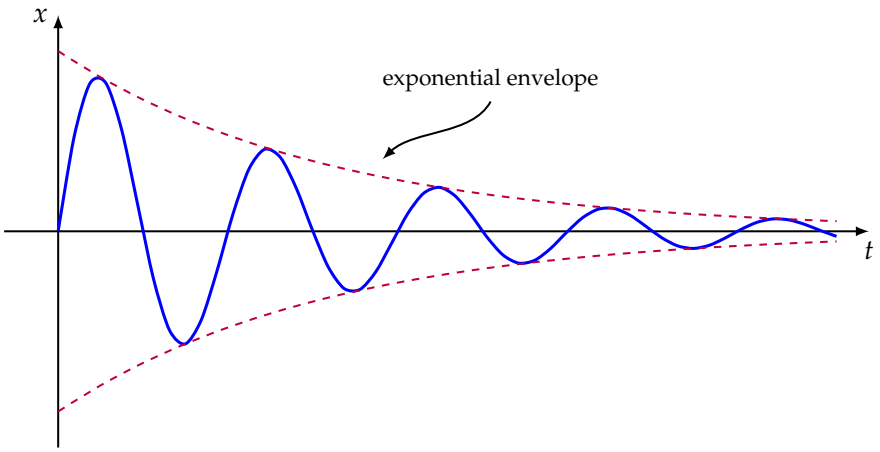
$E \propto x_0^2$, so doubling the amplitude gives four times the energy: $4 \times 20 = 80$ mJ.

Damping and resonance

Damped oscillations

In reality friction and resistive forces remove energy.

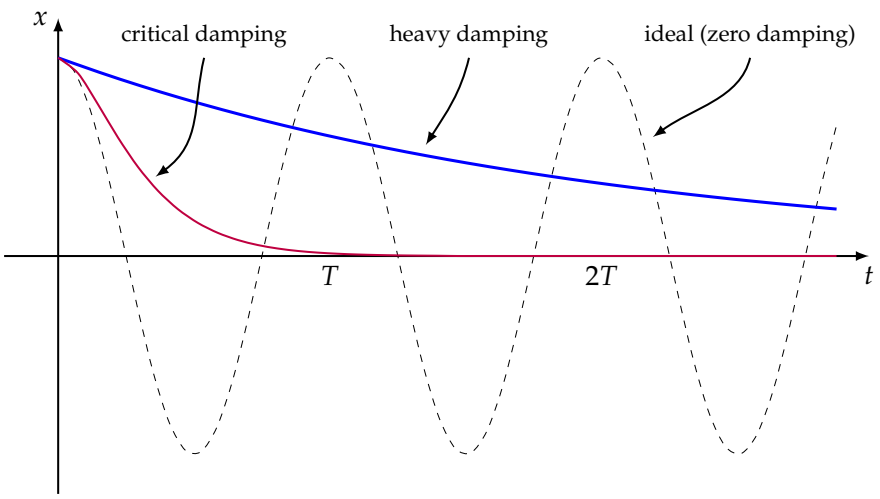
Damping is the decrease in amplitude of an oscillator because of energy loss to resistive forces.



Light damping: the amplitude decays exponentially.

- **light damping**: the amplitude falls gradually; the oscillator makes many oscillations; the period is almost unchanged
- **heavy damping**: so much resistance that there is no oscillation and the system returns to equilibrium very slowly
- **critical damping**: the border between them: the system returns to equilibrium in the shortest time without oscillating[3]

[3] Some boards define critical damping only by 'shortest time'; A-level exam boards insist on 'no oscillation'.



Displacement for different amounts of damping.

Critical damping is wanted in door closers, vehicle shock absorbers and artillery.

Example 20.7 A 250 g block on a spring ($k = 1.6$ N cm^{-1}) is displaced 5.0 cm and released. (a) Find ω. (b) Find the initial vibrational energy. (c) After some oscillations 40% of the energy has been lost to damping. Find the new amplitude.

(a) $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{160}{0.25}} \approx 25.3$ rad s^{-1}
(b) $E = \frac{1}{2}kx_0^2 = \frac{1}{2} \times 160 \times 0.050^2 = 0.20$ J
(c) $E \propto x_0^2$ and 60% remains: $x_0' = \sqrt{0.6}\, x_0 = \sqrt{0.6} \times 5.0 \approx 3.9$ cm

Question 20.5 A 360 g toy boat floats on water. It is pushed down and released. In the first four cycles its amplitude falls from 5.0 cm to 2.0 cm in 6.0 s. Find the energy loss.

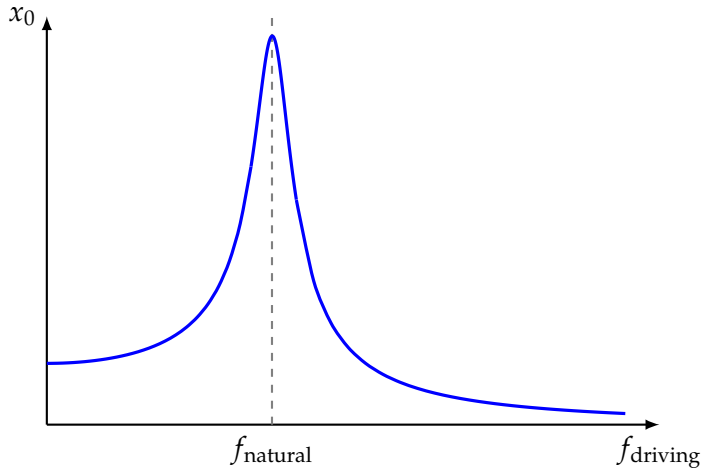
$T = \dfrac{6.0}{4} = 1.5$ s, so $\omega = \dfrac{2\pi}{1.5} = 4.19$ rad s^{-1}
$\Delta E = \frac{1}{2}m\omega^2(x_{01}^2 - x_{02}^2) = \frac{1}{2} \times 0.36 \times 4.19^2 \times (0.050^2 - 0.020^2) \approx 6.6 \times 10^{-3}$ J

Free and forced oscillations

- **free oscillation**: no gain or loss of energy; constant amplitude; frequency is the natural frequency
- **forced oscillation**: moves under an external driving force; after a long time its frequency tends to the driving frequency

Resonance

Resonance: when the driving frequency equals the system's natural frequency, the amplitude of the oscillation is a maximum.



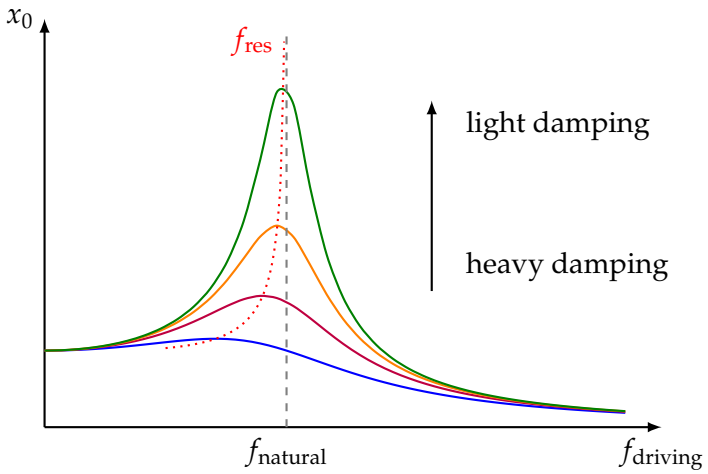
Amplitude against driving frequency (undamped: the peak tends to infinity).

• useful: microwave ovens (water molecules), MRI (nuclear precession), radio tuning circuits

• harmful: buildings in earthquakes, car suspension on bumps, bridges and skyscrapers in wind

Damping and resonance

Damping lowers the amplitude at __all__ frequencies; heavier damping makes the resonance peak __flatter__ and shifts it slightly to a __lower__ frequency. Engineering systems are often deliberately damped to reduce resonance.



Resonance curves for different amounts of damping.