Kinematics

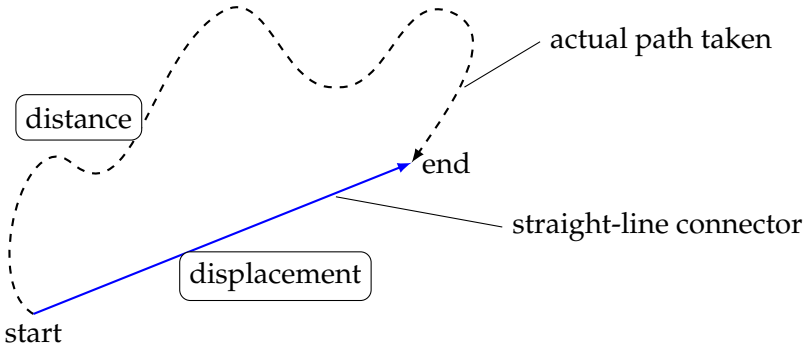
Name _____ Date _____

By the end of this topic I can…

- distinguish distance from displacement, and speed from velocity, and define acceleration
- read gradients and areas on displacement–time and velocity–time graphs
- use the equations of motion for constant acceleration, including free fall
- analyse projectile motion by treating the horizontal and vertical motion separately

Kinematic quantities

Distance and displacement



The path taken (dashed) and the displacement (straight arrow).

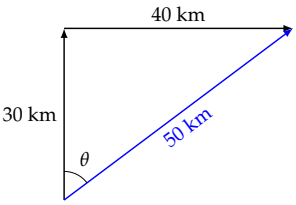
Displacement is the distance moved in a specific direction : the straight-line distance from the start to the end.

• **distance** is a scalar (size only); **displacement** is a vector (size and direction)

• both are measured in metres, but displacement can be zero for a journey with a long distance

Example 2.1 An athlete is running around a circular track of radius 60 m. When he completes one lap, what is the distance moved out? What about his displacement?

Distance moved is the perimeter of the circle: $s = 2\pi r = 2\pi \times 60 \approx 380$ m
athlete returns to same starting point after one lap, so displacement is zero



Example 2.2 A ship travels 30 km north, takes a right, and then travels 40 km east to reach its destination. Compare the distance and the displacement travelled.

Summing of all lengths gives distance: $30 + 40 = 70$ km
The displacement vector is shown on the diagram.
The magnitude of displacement $= \sqrt{30^2 + 40^2} = 50$ km
at an angle of $\theta = \tan^{-1}\frac{40}{30} \approx 53°$ east of north

Speed and velocity

Velocity is the change in displacement per unit time :

$$v = \frac{\Delta s}{\Delta t} \qquad [v] = \text{m s}^{-1}$$

- velocity is a vector; in one dimension we give direction with a sign : a negative velocity means motion in the opposite direction
- **average velocity**:

$$\text{average velocity} = \frac{\text{total displacement}}{\text{total time}}$$

- **average speed**:

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}$$

- **instantaneous velocity** is the average velocity over a very small Δt ($\Delta t \to 0$):

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}$$

Example 2.3 A cyclist travels a distance of 3.0 km in 20 minutes. She rests for 15 minutes. She then covers a further distance of 5.1 km in a time of 40 minutes. Calculate the average speed of the cyclist in m s^{-1}: (a) during the first 20 minutes of the journey, (b) for the whole journey.

For the first 20 minutes: $v = \frac{3.0\times10^3}{20\times60} = 2.5$ m s^{-1}
For whole journey: $v = \frac{(3.0+0+5.1)\times10^3}{(20+15+40)\times60} = 1.8$ m s^{-1}

Example 2.4 A man walks along a straight road for a distance of 800 m in 5.0 minutes. He then turns around, and walks along the same road for a distance of 280 m in 3.0 minutes. What is the average speed and the average velocity of this man during the 8.0 minutes?

Total distance travelled $= 800 + 280 = 1080$m, so average speed: $v = \frac{1080}{8.0\times60} = 2.25$ m s^{-1}
change of displacement $= 800 + (-280) = 520$m, so average velocity: $v = \frac{520}{8.0\times60} \approx 1.08$ m s^{-1}

Acceleration

Acceleration is the change in velocity per unit time:

$$a = \frac{\Delta v}{\Delta t} \qquad [a] = \text{m s}^{-2}$$

• acceleration is a vector. In linear motion take the direction of the initial velocity as positive

• $a > 0$: speeding up in the positive direction; $a < 0$: slowing down (deceleration)

• velocity changes if the speed changes or the direction changes, so an object moving in a curve always has a non-zero acceleration

• **instantaneous acceleration**:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

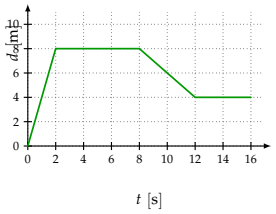
Example 2.5 A ball hits a barrier at right angles with a speed of 15 m s^{-1}. It makes contact with the barrier for 30 ms and then rebounds with a speed of 12 m s^{-1}. What is the average acceleration during the time of contact?

note that direction of velocity changed during rebound, so $\Delta v = 15 - (-12) = 27$ m s^{-1}
average acceleration: $a = \frac{\Delta v}{\Delta t} = \frac{27}{30\times10^{-3}} = 900$ m s^{-2}

Question 2.1 Is it possible for an object moving at a steady speed to have acceleration?

Yes. Acceleration is the rate of change of *velocity*, a vector. At a steady speed the direction can still change, for example a car going round a bend or a satellite in orbit, so the velocity changes and there is an acceleration.

Motion graphs



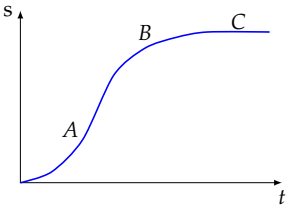
A generic displacement–time graph.

Displacement–time graphs

The gradient (tangent) of an s–t graph is the rate of change of displacement:

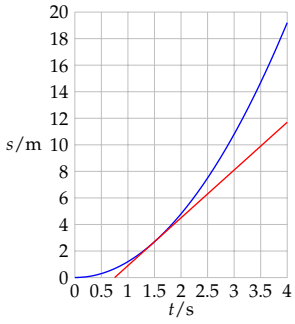
velocity = gradient of the s–t graph

- a straight line means constant velocity; a horizontal line means the object is at rest
- a curve means the velocity is changing: the steeper the line, the greater the speed
- a downward-sloping tangent means a negative velocity



Example 2.6 Describing the motion from the displacement-time graph shown.

Stage A: the gradient is increasing, so the object is speeding up.
Stage B: the gradient decreases, so it gradually slows down.
Stage C: the curve is horizontal, the gradient is zero, so the object has come to a stop.



Example 2.7 The diagram shows the displacement-time graph for a vehicle travelling along a straight road. Use the graph to find, (a) the average velocity during the first 4.0 s of the motion, (b) the velocity of the vehicle *at* time $t = 1.5$ s.

during first 4.0 s, average velocity is

$$v = \frac{\Delta s}{\Delta t} = \frac{19.2}{4.0} \approx 4.8 \text{ m s}^{-1}$$

to find velocity at $t = 1.5$ s, a tangent is drawn

gradient of tangent gives instantaneous velocity:

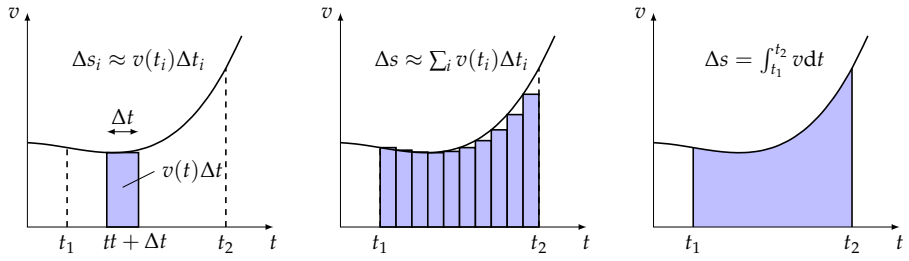
$$v = \frac{11.6-0}{4.0-0.75} \approx 3.6 \text{ m s}^{-1}$$

Velocity–time graphs

The gradient of a v–t graph gives the acceleration, and the area under it gives the change in displacement:

acceleration = gradient of the v–t graph

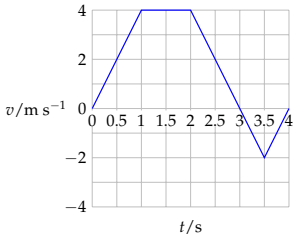
change in displacement = area under the v–t graph



(a) $\Delta s \approx v\,\Delta t$ (b) sum of many strips (c) area under graph

In a short interval v is almost constant, so $\Delta s \approx v\Delta t$ is the area of a thin strip. Add up all the strips and you get the area under the curve.

- a horizontal line: constant velocity (zero acceleration)
- a straight sloping line: constant acceleration
- area *below* the time axis counts as negative displacement (motion in the reverse direction)

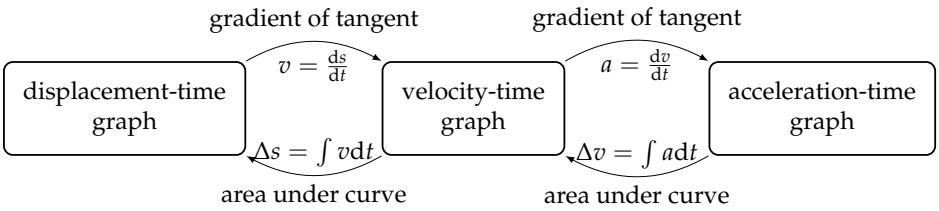


Example 2.8 The velocity of a toy car is shown. For the journey shown on the graph, use the graph to find (a) the total distance travelled, and (b) the total displacement travelled.

Distance is estimated using area under v-t graph
$0\sim3.0$ s: $s_1 = \frac{1}{2} \times 1.0 + 3.0 \times 4.0 = 8.0$ m
$3.0\sim4.0$ s: $s_2 = \frac{1}{2} \times 1.0 \times 2.0 = 1.0$ m
total distance $= 8.0 + 1.0 = 9.0$ m
total displacement $= (+8.0) + (-1.0) = 7.0$ m

Acceleration–time graphs

The area under an a–t graph is the change in velocity.



How the three graphs are linked: gradients go one way, areas the other.

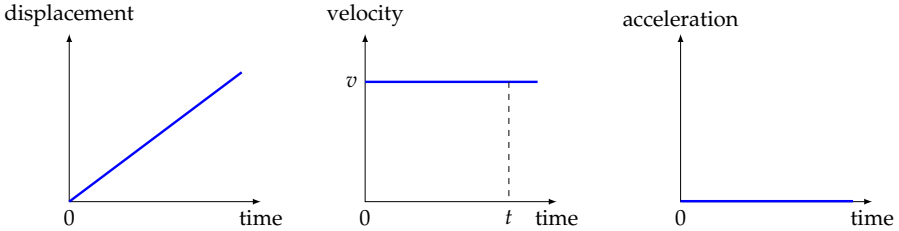
Constant velocity and constant acceleration

Constant velocity

For $v =$ constant we have $a =$ 0 and

$$s = vt$$

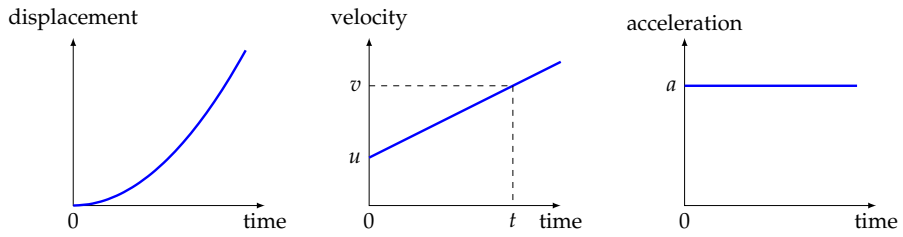
(from the origin; otherwise $s = s_0 + vt$). The s–t graph is a straight line.



Constant velocity: s–t, v–t and a–t.

Constant acceleration

Here a is constant, the v–t graph is a straight line and the s–t graph is a parabola.



Constant acceleration: s–t, v–t and a–t.

Equations of motion (SUVAT)

s = displacement, u = initial velocity, v = final velocity, a = acceleration, t = time.

- from $a = \dfrac{v-u}{t}$:

$$v = u + at$$

- the area under the v–t graph (average velocity $\times$ time):

$$s = \tfrac{1}{2}(u+v)t$$

- put the first into the second to remove v:

$$s = ut + \tfrac{1}{2}at^2$$

- eliminate t:

$$v^2 = u^2 + 2as$$

Using the equations: write down the five quantities, mark the three you know and the one you want, then pick the equation that has no mention of the fifth. They only apply when the acceleration is constant.

Example 2.9 A car starts from rest and accelerates uniformly at 5.0 m s^{-2} for 6.0 s.
(a) How fast is the car travelling at $t = 8.0$ s?
(b) What is the distance travelled by the car in this time?

$$v = u + at \quad \Rightarrow \quad v = 0 + 5.0 \times 6.0 = 30 \text{ m s}^{-1}$$
$$s = ut + \frac{1}{2}at^2 \quad \Rightarrow \quad s = 0 + \frac{1}{2} \times 5.0 \times 6.0^2 = 90 \text{ m}$$

Example 2.10 A car is travelling at 30 m s^{-1}. A hazard appears in front of the car, and the driver takes immediate action to stop the car. When brakes are applied, deceleration of the car is 5.0 m s^{-2}. What is the braking distance?

$$2as = v^2 - u^2 \quad \Rightarrow \quad s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 30^2}{2 \times (-5.0)} = 90 \text{ m}$$

Example 2.11 At the instant the traffic light turns green, a motorcycle waiting at the stop line starts with a constant acceleration of 2.0 m s^{-2}. At the same instant, a truck at a constant speed of 16 m s^{-1} overtakes and passes the motorcycle. How far beyond the stop line will the motorcycle overtake the truck?

Lets suppose overtake occurs at time t after motorcycle starts to accelerate, so the distance travelled by motorcycle:

$$s_m = u_m t + \frac{1}{2}at^2 \quad \Rightarrow \quad s_m = 0 + \frac{1}{2} \times 2.0 \times t^2$$

In the same time the distance travelled by truck:

$$s_t = v_t t \quad \Rightarrow \quad s_t = 16t$$

The overtake when:

$$s_m = s_t \quad \Rightarrow \quad \frac{1}{2} \times 2.0 \times t^2 = 16t \quad \Rightarrow \quad t = 16 \text{ s}$$

Substitute t into either s_m or s_t, one finds distance travelled: $s = 256$ m

Free fall and projectiles

Free fall

Ignoring air resistance, every object near the Earth's surface falls with the same constant acceleration $g \approx$ 9.81 m s^{-2} downwards, independent of its mass.

Released from rest ($u = 0$, $a = g$):

$v = gt$ $\qquad$ $s = \frac{1}{2}gt^2$

Example 2.12 An object is released from rest from a height of $h = 24$ m and falls freely under gravity. Air resistance is negligible.
(a) How long does it take to hit the ground?
(b) What is its speed when hitting the ground?

$$h = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 24}{9.81}} \approx 2.21 \text{ s}$$

$$v = gt = 9.81 \times 2.21 \quad \Rightarrow \quad v \approx 21.7 \text{ m s}^{-1}$$

to find final velocity, one can also use $v^2 = u^2 + 2as$:

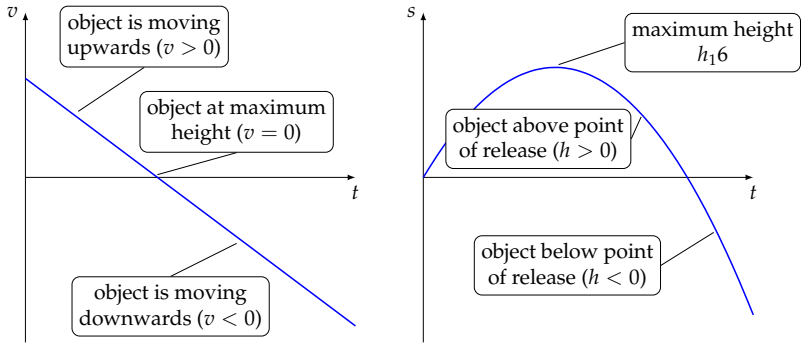
$$v^2 = \cancel{u^2}^{0} + 2gh = 2 \times 9.81 \times 24 \quad \Rightarrow \quad v \approx 21.7 \text{ m s}^{-1}$$

Upward projection

Take upwards as positive, so $a =$ -g :

$v = u - gt$ $\qquad$ $s = ut - \frac{1}{2}gt^2$

- $v > 0$: moving up; $v < 0$: moving down; at the highest point $v =$ 0
- $s > 0$: above the point of release; $s < 0$: below it
- the time up equals the time down back to the same height, and the speed on return equals the initial speed



v–t and s–t graphs for an object thrown upwards.

Example 2.13 A ball is projected vertically upwards at 12 m s^{-1}. Air resistance is negligible. (a) Find the time taken for the ball to reach the highest position. (b) Find the greatest height.

maximum height is reached when $v = 0$, so

$$v = u - gt = 0 \quad \Rightarrow \quad t = \frac{u}{g} = \frac{12}{9.81} \approx 1.22 \text{ s}$$

$$H_{\max} = ut - \tfrac{1}{2}gt^2 = 12 \times 1.22 - \tfrac{1}{2} \times 9.81 \times 1.22^2 \approx 7.34 \text{ m}$$

It is also possible to use $v^2 - u^2 = 2as$ to find $H_{\max}$:

$$0^2 - u^2 = 2(-g)H_{\max} \quad \Rightarrow \quad H_{\max} = \frac{u^2}{2g} = \frac{12^2}{2 \times 9.81} \approx 7.34 \text{ m}$$

Example 2.14 A stone is thrown vertically upwards with an initial velocity of 14.0 m s^{-1} from the edge of a cliff that is 35 m from the sea below.
(a) Find the speed at which it hits the sea.
(b) Find the time taken for the stone to hit the sea.

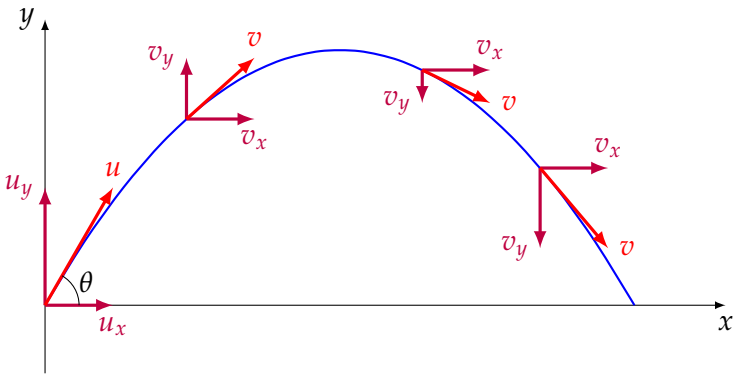
Take positive direction to point upwards, we use $v^2 - u^2 = 2as$ to find $v^2 = 14.0^2 + 2 \times (-9.81) \times (-35) \approx 883 \text{ m}^2 \text{ s}^{-2} \quad \Rightarrow \quad v \approx -29.7 \text{ m s}^{-1}$. To find time, we can use $v = u - gt$, hence: $t = \frac{v-u}{-g} = \frac{-29.7-14.0}{-9.81} \approx 4.46$ s.
One can also attempt $s = ut - \frac{1}{2}gt^2$, this leads to the equation: $-35 = 14.0t - \frac{1}{2} \times 9.81t^2$.
This quadratic equation in t gives two roots: $t_1 \approx 4.46$ s, and $t_2 \approx -1.60$ s.
Note that a negative root should be discarded since it means stones hits the sea before it is thrown so time taken for stone to hit the sea is $t \approx 4.46$ s

Projectile motion

A projectile moves under gravity alone. Its horizontal and vertical motions are ___independent___ :

- horizontally: $a_x =$ ___0___, so the horizontal velocity is ___constant___
- vertically: $a_y =$ ___-g___ (upwards positive), constant acceleration

The path is a ___parabola___ .



Velocity components at different points of a projectile's path.

Launched at speed u at angle θ above the horizontal, $u_x = u\cos\theta$ and $u_y = u\sin\theta$:

$v_x = u\cos\theta$ $\quad$ $x = u\cos\theta\, t$

$v_y = u\sin\theta - gt$ $\quad$ $y = u\sin\theta\, t - \frac{1}{2}gt^2$

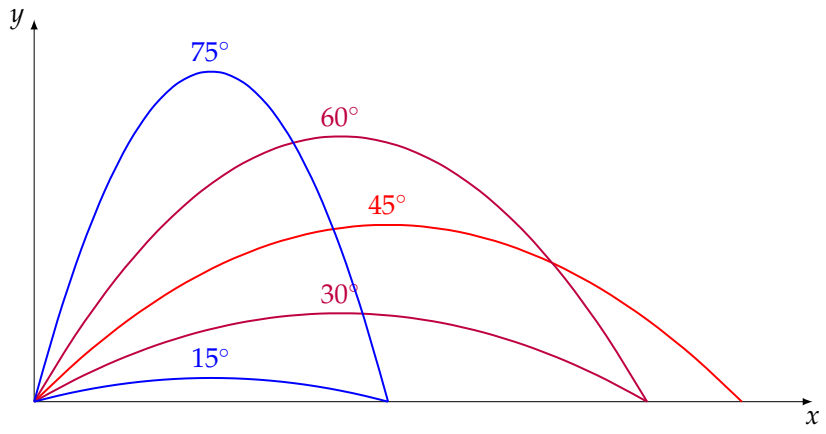
The resultant speed is $v = \sqrt{v_x^2 + v_y^2}$, at angle $\tan^{-1}(v_y/v_x)$ to the horizontal.

- at the highest point $v_y =$ 0, so the time to the top is $t = \dfrac{u\sin\theta}{g}$ and

$$H_{\max} = \frac{u^2\sin^2\theta}{2g}$$

- back at launch height $y = 0$, so the total time of flight and the range are

$$T = \frac{2u\sin\theta}{g} \qquad X = \frac{u^2\sin 2\theta}{g}$$



Trajectories for the same launch speed at different angles.

The range is greatest at an angle of 45° (for the same launch and landing height); the time of flight is greatest for $\theta = 90°$. Angles that add to 90° give the same range.

Example 2.15 A ball is thrown from a point O at 15 m s^{-1} at an angle of 40° to the horizontal. The ball reaches its highest position at point P. Ignore the effects of air resistance. (a) How long does it take to reach P? (b) What is the magnitude of the displacement OP?

At the highest point: $v_y = u_y - gt = 0 \quad \Rightarrow \quad t = \frac{u\sin\theta}{g} = \frac{15\times\sin 40^\circ}{9.81} \approx 0.983$ s
Vertical displacement: $y = u_y t - \frac{1}{2}gt^2 = 15\sin 40^\circ \times 0.983 - \frac{1}{2} \times 9.81 \times 0.983^2 \approx 4.74$ m
Horizontal displacement: $x = u_x t = 15\cos 40^\circ \times 0.983 \approx 11.3$ m
Resultant displacement: $|OP| = \sqrt{x^2 + y^2} = \sqrt{11.3^2 + 4.74^2} \approx 12.2$ m

Example 2.16 A small object is horizontally projected at 7.20 ms^{-1} from a surface at a height of $h = 1.2$ m above the ground. Assume there is no air resistance. (a) What is the time taken for the object to hit the ground? (b) What is the horizontal range? (c) Find the velocity at which the object hits the ground.

Vertically, take downward as positive: $h = \cancel{u_y t}^{0} + \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\times 1.2}{9.81}} \approx 0.495$ s
Horizontal range: $x = u_x t = 7.20 \times 0.495 \approx 3.56$ m
Final vertical velocity: $v_y = \cancel{u_y}^{0} + gt = 9.81 \times 0.495 \approx 4.85$ m s^{-1}
Magnitude of resultant velocity: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{7.20^2 + 4.85^2} \approx 8.68$ m s^{-1}
Angle to which resultant velocity makes with horizontal: $\phi = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{4.85}{7.20} \approx 34^\circ$

Practice questions

Question 2.2 What is the distance covered for a car that travels half a lap along a circular path of radius of 200 m. What about the displacement?

$s = \pi r = \pi \times 200 \approx 6.3 \times 10^2$ m. The displacement is a diameter: $2r = 400$ m.

Question 2.3 A car travels 2400 m east in 3.0 minutes, then takes a left turn, and then travels 700 m north in 1.5 minutes. What is the average speed and the average velocity for this journey?

Distance $= 2400 + 700 = 3100$ m, time $= 180 + 90 = 270$ s, so average speed $= \dfrac{3100}{270} \approx 11.5$ m s^{-1}.
Displacement $= \sqrt{2400^2 + 700^2} = 2500$ m, so average velocity $= \dfrac{2500}{270} \approx 9.3$ m s^{-1} at $\tan^{-1} \dfrac{700}{2400} \approx 16^\circ$ north of east.

Question 2.4 A vehicle initially travels at a steady speed of 15 m s^{-1}. It accelerates uniformly for 10 s to reach a higher speed of 20 m s^{-1}. It maintains at this speed for 20 s, and then decelerates uniformly to a stop in the last 10 s. (a) Sketch the velocity-time graph for this motion. (b) Sketch the acceleration-time graph. (c) Find the distance travelled during the 40 s.

(a) v–t: a straight line rising from 15 m s^{-1} to 20 m s^{-1} in 10 s, a horizontal line at 20 m s^{-1} for 20 s, then a straight line falling to zero in 10 s.
(b) a–t: $a = \dfrac{20 - 15}{10} = +0.50$ m s^{-2} for 10 s, then 0 for 20 s, then $a = \dfrac{0 - 20}{10} = -2.0$ m s^{-2} for 10 s.
(c) Area under the v–t graph: $\frac{1}{2}(15 + 20) \times 10 + 20 \times 20 + \frac{1}{2} \times 20 \times 10 = 175 + 400 + 100 = 675$ m.

Question 2.5 A vehicle moving at 14 m s^{-1} accelerate uniformly to 26 m s^{-1} in 6.0 s. (a) What is the average velocity during this time? (b) What is the acceleration during this time (c) What is the distance travelled by the vehicle? (d) The vehicle then braked with constant deceleration to stop in another 8.0 s. What is the distance travelled during the time when brakes are applied?

(a) $\bar{v} = \frac{1}{2}(14 + 26) = 20$ m s^{-1}
(b) $a = \dfrac{26 - 14}{6.0} = 2.0$ m s^{-2}
(c) $s = \bar{v}t = 20 \times 6.0 = 120$ m
(d) $s = \frac{1}{2}(u + v)t = \frac{1}{2}(26 + 0) \times 8.0 = 104$ m

Question 2.6 Two balls are dropped from rest from the same height. The second ball is released 0.80 s after the first one. What is their separation 1.5 s after the second ball is dropped?

The first ball has been falling for $0.80 + 1.5 = 2.3$ s: $s_1 = \frac{1}{2} \times 9.81 \times 2.3^2 = 25.9$ m.
The second ball has fallen for 1.5 s: $s_2 = \frac{1}{2} \times 9.81 \times 1.5^2 = 11.0$ m.
Separation $= 25.9 - 11.0 \approx 14.9$ m.

Question 2.7 The acceleration of free fall on Pluto is about one-fifteenth of that on Earth. If it takes a time of T for a rock to fall from rest a distance of S, what is the time taken, in terms of T, for a rock to fall from rest through the same distance S on Pluto?

$S = \frac{1}{2}gt^2$, so $t = \sqrt{2S/g} \propto g^{-1/2}$. With $g_P = g/15$: $t_P = \sqrt{15}\,T \approx 3.9\,T$.

Question 2.8 Someone wants to toss a ball onto a platform that is at a height of 20 m above him. What is the minimum initial velocity needed to launch the ball?

At the platform, at best, $v = 0$: $v^2 = u^2 + 2as \quad \Rightarrow \quad 0 = u^2 - 2 \times 9.81 \times 20 \quad \Rightarrow \quad u = \sqrt{392} \approx 19.8$ m s^{-1}.

Question 2.9 A ball rolls off a table and lands at a position of a horizontal distance of 1.2 m from the table. The table is 0.95 m high. Find the speed at which the ball leaves the table.

Vertically (from rest): $0.95 = \frac{1}{2} \times 9.81\,t^2 \quad \Rightarrow \quad t = \sqrt{\dfrac{2 \times 0.95}{9.81}} = 0.44$ s.

Horizontally the velocity is constant: $u = \dfrac{x}{t} = \dfrac{1.2}{0.44} \approx 2.7$ m s^{-1}.

Question 2.10 A ball is kicked from the ground towards a vertical barrier. The barrier is at a horizontal distance of 18 m from the initial position of the ball. The ball strikes the barrier after 1.5 s at a height of 2.5 m above the ground. (a) Find the magnitude and the direction of the initial velocity. (b) Find the magnitude and the direction of the velocity at which the ball hits the barrier.

(a) Horizontal: $u_x = \dfrac{18}{1.5} = 12.0\ \text{m s}^{-1}$. Vertical: $2.5 = u_y(1.5) - \frac{1}{2}(9.81)(1.5)^2 \Rightarrow u_y = 9.02\ \text{m s}^{-1}$.
$u = \sqrt{12.0^2 + 9.02^2} \approx 15\ \text{m s}^{-1}$ at $\tan^{-1}\dfrac{9.02}{12.0} \approx 37^\circ$ above the horizontal.
(b) $v_x = 12.0\ \text{m s}^{-1}$, $v_y = 9.02 - 9.81 \times 1.5 = -5.69\ \text{m s}^{-1}$ (moving downwards).
$v = \sqrt{12.0^2 + 5.69^2} \approx 13\ \text{m s}^{-1}$ at $\tan^{-1}\dfrac{5.69}{12.0} \approx 25^\circ$ below the horizontal.