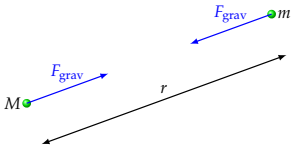
Gravitational Fields

Name _____ Date _____

By the end of this topic I can...
- use Newton's law of gravitation, and apply it to circular orbits and Kepler's third law
- explain apparent weight and the effect of the Earth's rotation
- define field strength and potential, and use $g = GM/r^2$ and $\varphi = -GM/r$
- use $E_p = -GMm/r$ with energy conservation, including escape speed

Gravitational forces

Newton's law of gravitation



Gravitational attraction between M and m.

Any object with mass attracts any other massive object through the gravitational force.

Newton's law of gravitation: the gravitational force between two point masses is proportional to the product of their masses and inversely proportional to the square of their distance.

This law appeared in Newton's *Principia* (1687). In symbols:

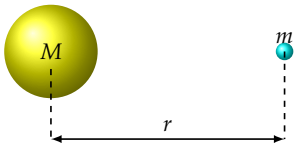
$$F_{\text{grav}} = \frac{GMm}{r^2}$$

where $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} is the gravitational constant.

• gravitational force is always attractive, and acts between any two masses (it is universal)

• the law refers to point masses: particles with no size, so the distance r is well defined

A sphere with uniform mass distribution (a star or planet) can be treated as a point mass at its centre, so r is the distance between centres.[1]



A uniform sphere acts as a point mass at its centre.

Example 19.1 The Earth is a uniform sphere of radius $R = 6.4 \times 10^6$ m and mass $M = 6.0 \times 10^{24}$ kg. Estimate the gravitational force on a 60 kg man at sea level.

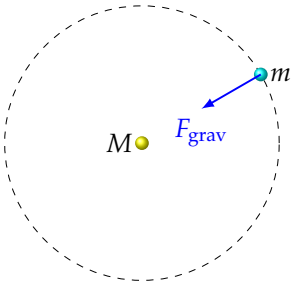
$$F = \frac{GMm}{R^2} = \frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 60}{(6.4 \times 10^6)^2} \approx 586 \text{ N}$$

[1] This is the *shell theorem*: a spherically symmetric shell attracts outside objects as if all its mass were at its centre, and exerts no net force on anything inside it.

Question 19.1 Estimate the gravitational force between you and the person nearest to you as you read this.

Taking $m_1 \approx 60$ kg, $m_2 \approx 70$ kg and $r \approx 1$ m:

$$F = \frac{6.67 \times 10^{-11} \times 60 \times 70}{1^2} \approx 3 \times 10^{-7} \text{ N}$$



A satellite in circular orbit.

Planetary motion

A planet or satellite moves in a circular orbit when gravity provides the centripetal force :

$$F_{grav} = F_c \quad \Rightarrow \quad \boxed{\frac{GMm}{r^2} = \frac{mv^2}{r}}$$

or, using $v = \omega r$:

$$\boxed{\frac{GMm}{r^2} = m\omega^2 r}$$

Example 19.2 GPS satellites move in circular orbits about 20 000 km above the Earth's surface. ($R = 6.4 \times 10^6$ m, $M = 6.0 \times 10^{24}$ kg.) (a) Find the speed of a GPS satellite. (b) Find its orbital period.

(a) $\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6 + 2.0 \times 10^7}} \approx 3.9 \times 10^3$ m s^{-1}

(b) $T = \frac{2\pi r}{v} = \frac{2\pi \times (6.4 \times 10^6 + 2.0 \times 10^7)}{3.9 \times 10^3} \approx 4.3 \times 10^4$ s

Example 19.3 A **geostationary satellite** stays above the same point on the equator, so its period is 24 hours. Find the radius of its orbit.

$$\frac{GMm}{r^2} = m\omega^2 r = m\left(\frac{2\pi}{T}\right)^2 r \quad \Rightarrow \quad r^3 = \frac{GMT^2}{4\pi^2}$$

$$r = \left(\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times (24 \times 3600)^2}{4\pi^2}\right)^{1/3} \approx 4.23 \times 10^7 \text{ m}$$

Example 19.4 Assuming circular orbits, show that for the planets the square of the orbital period is proportional to the cube of the orbital radius.[2]

$$\frac{GMm}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r \quad \Rightarrow \quad T^2 = \frac{4\pi^2}{GM} r^3$$

G and the Sun's mass M are constants, so $\frac{4\pi^2}{GM}$ is constant: $T^2 \propto r^3$.

[2] Kepler's third law (early 1600s). Newton showed that all three of Kepler's laws follow from his law of gravitation, and $T^2 \propto r^3$ holds for elliptical orbits too.

Question 19.2 Light takes about 8.0 minutes to travel from the Sun to the Earth. (a) What is the mass of the Sun? (b) At what speed does the Earth move round the Sun?

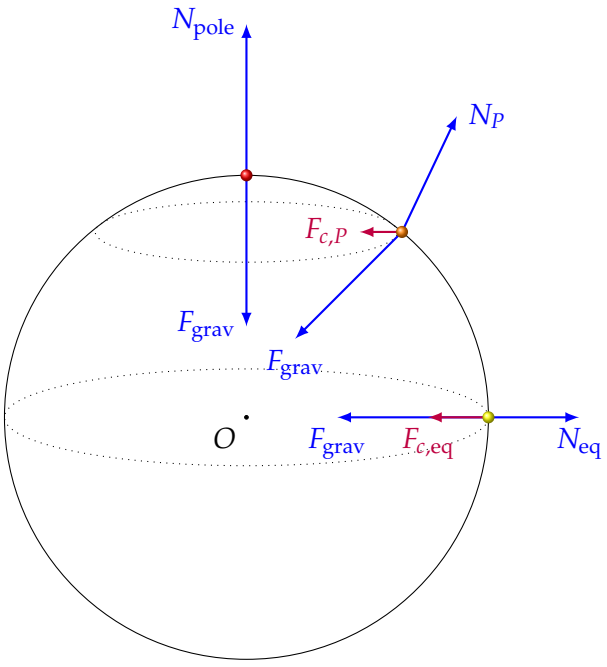
$r = ct = 3.0 \times 10^8 \times 480 = 1.44 \times 10^{11}$ m, and $T = 1$ year $= 3.16 \times 10^7$ s.

(a) $M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2 \times (1.44 \times 10^{11})^3}{6.67 \times 10^{-11} \times (3.16 \times 10^7)^2} \approx 1.8 \times 10^{30}$ kg

(b) $v = \frac{2\pi r}{T} = \frac{2\pi \times 1.44 \times 10^{11}}{3.16 \times 10^7} \approx 2.9 \times 10^4$ m s^{-1}

Apparent weight

The actual weight of an object is the gravitational force on it. Its apparent weight is the upward contact force (from the ground or a spring balance) that opposes gravity and stops it falling. The two differ when there is vertical acceleration, or buoyancy, or because of the Earth's rotation.



Apparent weight at the equator, at latitude P and at the pole (not to scale).

An object on the ground rotates with the Earth, so the resultant of gravity and the

contact force provides the centripetal force. At the equator:

$$F_{\text{grav}} - N_{\text{eq}} = m\omega^2 R \quad \Rightarrow \quad N_{\text{eq}} = \frac{GMm}{R^2} - m\omega^2 R$$

At the poles there is no circular motion, so $N_{\text{pole}} =$ $\frac{GMm}{R^2}$.

Nearer the equator the circle is larger, so more centripetal force is needed and the apparent weight is smaller.

Example 19.5 A 5.0 kg stone hangs from a newton-meter at the equator. ($R = 6370$ km, $M = 5.97 \times 10^{24}$ kg.) (a) What is the gravitational force on the stone? (b) What does the meter read?

(a) $F_{\text{grav}} = \dfrac{GMm}{R^2} = \dfrac{6.67 \times 10^{-11} \times 5.97 \times 10^{24} \times 5.0}{(6.37 \times 10^6)^2} \approx 49.07$ N

(b) $F_c = m\omega^2 R = 5.0 \times \dfrac{4\pi^2}{(24 \times 3600)^2} \times 6.37 \times 10^6 \approx 0.17$ N

reading $= F_{\text{grav}} - F_c = 49.07 - 0.17 \approx 48.90$ N

Question 19.3 Why are astronauts in a space station said to be *weightless*?

The station and the astronauts are both in free fall with the same acceleration (gravity still acts, and is about 90% of its surface value). There is no contact force between them, so the apparent weight is zero.

Question 19.4 How do you find the apparent weight of an object at an arbitrary latitude P? Does it act vertically downwards? Give your reasons.

The object moves in a circle of radius $R\cos\lambda$ about the Earth's axis, so it needs a centripetal force $m\omega^2 R\cos\lambda$ directed towards the axis. The apparent weight is the vector difference of F_{grav} (towards the Earth's centre) and this centripetal force.
These two vectors are not in line, except at the equator and the poles, so the apparent weight does not point to the Earth's centre: it is not exactly vertical.

Gravitational fields

To explain how masses attract at a distance we use the idea of a force field.

A gravitational field is a region of space where a mass experiences a force.

Any mass M produces a field around it; a small test mass m placed in the field feels a force. We describe the field with field strength (to find forces) and potential (to find energies).

Gravitational field strength

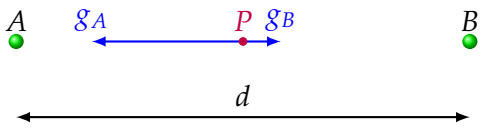
Gravitational field strength is the gravitational force per unit mass: $g = \frac{F_{\text{grav}}}{m}$

- unit: $[g] =$ N kg^{-1}, which is the same as m s^{-2}
- for an isolated source M, at distance r a test mass feels $F = \frac{GMm}{r^2}$, so

$$g = \frac{GM}{r^2}$$

g depends on the source mass M, not on the test mass m.

- g is a vector and points towards the source (gravity is attractive). With several sources, add the contributions as vectors.



Example 19.6 Star A of mass 6.0×10^{30} kg and star B of mass 1.5×10^{30} kg are 2.0×10^{12} m apart. (a) What is the field strength at the mid-point P? (b) What force acts on a 4.0×10^{6} kg comet at P?

(a) g_A points towards A and g_B towards B: opposite directions.

$$g_P = g_A - g_B = \frac{GM_A}{r^2} - \frac{GM_B}{r^2}$$

$$= 6.67 \times 10^{-11} \frac{6.0 \times 10^{30} - 1.5 \times 10^{30}}{(1.0 \times 10^{12})^2} \approx 3.0 \times 10^{-4} \text{ N kg}^{-1}$$

(b) $F = mg = 4.0 \times 10^{6} \times 3.0 \times 10^{-4} \approx 1.2 \times 10^{3}$ N

Acceleration of free fall

If gravity is the only force, $F_{\text{net}} = F_{\text{grav}}$, so $ma = mg$:

$$a = g$$

Gravitational field strength is the acceleration of free fall.[3]

Example 19.7 The Earth has radius 6370 km. (a) Find the mass of the Earth. (b) Find the acceleration of free fall at the top of Mount Everest (height 8.8 km).[4]

(a) at the surface $g_s = \dfrac{GM}{R^2} \Rightarrow 9.81 = \dfrac{6.67 \times 10^{-11} \times M}{(6.37 \times 10^6)^2} \Rightarrow M \approx 5.97 \times 10^{24}$ kg

(b) $g = \dfrac{GM}{(R+H)^2} = \dfrac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{(6.37 \times 10^6 + 8.8 \times 10^3)^2} \approx 9.79 \text{ N kg}^{-1}$

[3] Strictly, the two m's are different: inertial mass (resistance to acceleration) and gravitational mass. No experiment has found any difference between them. Einstein's equivalence principle, the root of general relativity, says a uniform acceleration and a uniform gravitational field cannot be told apart.

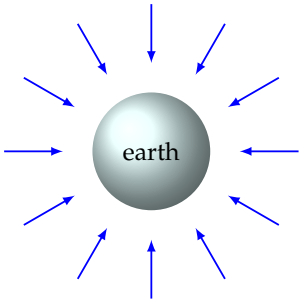
[4] Cavendish measured G in 1798 and could then 'weigh the world'.

Gravitational field lines

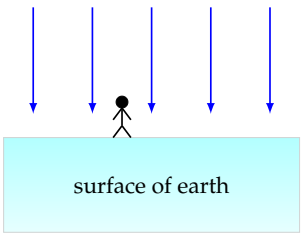
Field lines show the pattern of field strength.

- the direction of the lines is the direction of the field
- the spacing of the lines shows the strength: closer lines mean a stronger field
- gravitational field lines always end on a mass (gravity is attractive)

Around a planet the field is radial. Near its surface the lines are parallel and equally spaced: a uniform field.



Radial field around the Earth.



Near the surface the field is almost uniform.

Gravitational potential and potential energy

Potential energy

Potential energy is the energy an object has because of its position in a force field. Work done by the field decreases P.E.; work done against the field increases P.E.:

$$W = -\Delta E_p$$

To define P.E. at a point X: choose a position where $E_p = 0$, find the work W done by the field moving the object from there to X; then $E_{p,X} = -W$.

Near the Earth's surface take $E_p = 0$ at sea level. Lifting m through height h means work $-mgh$ is done by gravity, so

$$\Delta E_p = mg\,\Delta h \quad \text{and} \quad E_p = mgh$$

Gravitational potential energy



We choose $E_p = 0$ at $r = \infty$ (no force, so no energy).

Gravitational potential energy is the work done by gravity bringing a mass to a point from infinity.

The force $\frac{GMm}{x^2}$ changes with x, so the work is the area under the F–x graph:[5]

$W = \frac{GMm}{r}$ and $\Delta E_p = -W$, with $E_p(\infty) = 0$, so

$$E_p = -\frac{GMm}{r}$$

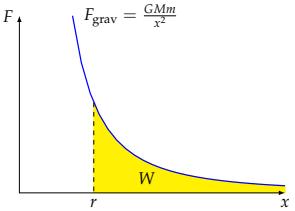
- $E_p \to 0$ as $r \to \infty$, matching our choice of zero
- E_p is a scalar; it is always negative because gravity is attractive
- separating masses means doing work, so E_p increases with r; the maximum, at infinity, is zero
- $E_p = mgh$ only works near the surface (uniform field); $E_p = -GMm/r$ is the general formula [6]

Example 19.8 A meteor is moving towards a planet of mass M. At distance r_1 its speed is v_1; at r_2 it is v_2. Only gravity acts. Relate these quantities.

Energy is conserved: K.E. + G.P.E. = constant

$$\tfrac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \tfrac{1}{2}mv_2^2 - \frac{GMm}{r_2}$$

[5] In general $W = \int F\,dx$. Here $W = \int_r^\infty \frac{GMm}{x^2}\,dx = \left[-\frac{GMm}{x}\right]_r^\infty = \frac{GMm}{r}$.



F–x graph: the shaded area is the work W.

[6] For $r_1 \approx r_2 \approx R$: $\Delta E_p = GMm\frac{r_1 - r_2}{r_1 r_2} \approx m\frac{GM}{R^2}\Delta r = mg\,\Delta h$.

Example 19.9 An object thrown fast enough from a planet's surface escapes its gravity. The minimum speed needed is the **escape velocity**. Find it for the Earth (use $R = 6.4 \times 10^6$ m, $M = 6.0 \times 10^{24}$ kg).

Energy at the surface = energy at infinity (the minimum case has zero K.E. there):

$$\tfrac{1}{2}mu^2 - \frac{GMm}{R} = \tfrac{1}{2}mv^2 + 0 \quad \text{with } v \geq 0 \;\Rightarrow\; u^2 \geq \frac{2GM}{R}$$

$$u_{\min} = \sqrt{\frac{2GM}{R}}$$

$$u_{\min} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6}} \approx 1.12 \times 10^4 \text{ m s}^{-1}$$

Question 19.5 A rock falls from rest from a height $3R$ above the surface of an airless planet of uniform density, radius R and mass M. Show that it hits the ground at speed $v = \sqrt{\dfrac{3GM}{2R}}$.

It starts at $r = 4R$ and ends at $r = R$. Gain in K.E. = loss in G.P.E.:

$$\tfrac{1}{2}mv^2 = \frac{GMm}{R} - \frac{GMm}{4R} = \frac{3GMm}{4R} \quad \Rightarrow \quad v = \sqrt{\frac{3GM}{2R}}$$

Question 19.6 A probe of mass m orbits a planet of mass M in a circle of radius r. (a) Show that its total mechanical energy is $E_{\text{total}} = -\dfrac{GMm}{2r}$. (b) If small resistive forces act, state the change in its orbital radius and speed.

(a) $\dfrac{GMm}{r^2} = \dfrac{mv^2}{r} \quad \Rightarrow \quad \text{K.E.} = \tfrac{1}{2}mv^2 = \dfrac{GMm}{2r}$

$E_{\text{total}} = \dfrac{GMm}{2r} - \dfrac{GMm}{r} = -\dfrac{GMm}{2r}$

(b) Energy is lost, so E_{total} becomes more negative: the radius **decreases**, and since $v^2 = GM/r$ the speed **increases**.

Question 19.7 A *black hole* is a region where gravity is so strong that not even light can escape. (a) Show that a star of mass M must be compressed below the Schwarzschild radius $R_S = \dfrac{2GM}{c^2}$ to become one. (b) Show that R_S for the Sun ($M = 2.0 \times 10^{30}$ kg) is about 3 km.[7]

(a) Light cannot escape if the escape speed exceeds c: $\sqrt{\dfrac{2GM}{R}} > c \quad \Rightarrow \quad R < \dfrac{2GM}{c^2}$

(b) $R_S = \dfrac{2 \times 6.67 \times 10^{-11} \times 2.0 \times 10^{30}}{(3.0 \times 10^8)^2} \approx 3.0 \times 10^3 \text{ m} = 3 \text{ km}$

[7] For very strong fields Newtonian gravity fails and general relativity is needed. That the Newtonian and relativistic radii agree is a coincidence.

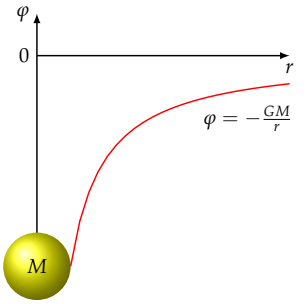
Gravitational potential

It is useful to define the potential energy ___per unit mass___: $\varphi = \dfrac{E_p}{m}$.

___Gravitational potential___ at a point is the ___work done by gravity___ bringing a ___unit mass___ from ___infinity___ to that point.

- unit: $[\varphi] =$ ___J kg^{-1}___
- for an isolated source M:

$$\varphi = -\frac{GM}{r}$$



Potential against distance from an isolated mass.

- $\varphi =$ ___0___ at infinity, and φ is always ___negative___; it increases with distance from the source
- φ is a ___scalar___: for several sources, add the potentials as ___scalars___
- a mass moving through a change in potential $\Delta\varphi$ gains kinetic energy:

$$\tfrac{1}{2}mv^2 = m\,\Delta\varphi$$

Example 19.10 Star A ($M_A = 1.5 \times 10^{30}$ kg) and planet B ($M_B = 6.0 \times 10^{26}$ kg) form an isolated system. Point P lies between them, $r_A = 2.0 \times 10^{12}$ m from A and $r_B = 8.0 \times 10^{10}$ m from B. (a) Find the potential at P. (b) A meteor starts at rest very far away. Find its speed at P.

(a) $\varphi_P = \varphi_A + \varphi_B = -6.67 \times 10^{-11}\left(\dfrac{1.5 \times 10^{30}}{2.0 \times 10^{12}} + \dfrac{6.0 \times 10^{26}}{8.0 \times 10^{10}}\right) \approx -5.05 \times 10^7$ J kg^{-1}

(b) gain in K.E. = loss in G.P.E.: $\frac{1}{2}v^2 = \varphi_\infty - \varphi_P = -\varphi_P$

$$v = \sqrt{2 \times 5.05 \times 10^7} \approx 1.01 \times 10^4 \text{ m s}^{-1}$$

Example 19.11 The Moon is an isolated sphere of radius 1.74×10^3 km, with potential -2.82×10^6 J kg^{-1} at its surface. (a) Find the Moon's mass. (b) A stone moves from $3R$ to $2R$ from the Moon's centre. Find the change in potential. (c) If it starts from rest, find its final speed.

(a) $\varphi(R) = -\dfrac{GM}{R} \quad \Rightarrow \quad M = \dfrac{2.82 \times 10^6 \times 1.74 \times 10^6}{6.67 \times 10^{-11}} \approx 7.36 \times 10^{22}$ kg

(b) $\Delta\varphi = -\dfrac{GM}{3R} + \dfrac{GM}{2R} = \dfrac{GM}{6R} = \dfrac{2.82 \times 10^6}{6} \approx 4.70 \times 10^5$ J kg^{-1} (a *decrease*)

(c) $\frac{1}{2}v^2 = \Delta\varphi \quad \Rightarrow \quad v = \sqrt{2 \times 4.70 \times 10^5} \approx 970$ m s^{-1}

Question 19.8 The Moon has radius 1700 km and mass 7.4×10^{22} kg. (a) Find the change in potential when an object rises from the surface to 800 km above it. (b) A rock is projected upwards at 1800 m s^{-1} from the surface. Find its speed at 800 km. (c) Can it escape completely?

(a) $\Delta\varphi = GM\left(\dfrac{1}{1.7 \times 10^6} - \dfrac{1}{2.5 \times 10^6}\right) \approx 9.3 \times 10^5$ J kg^{-1} (an increase)

(b) $\frac{1}{2}v^2 = \frac{1}{2}u^2 - \Delta\varphi \quad \Rightarrow \quad v^2 = 1800^2 - 2 \times 9.3 \times 10^5 \quad \Rightarrow \quad v \approx 1.2 \times 10^3$ m s^{-1}

(c) potential at the surface $= -GM/R \approx -2.9 \times 10^6$ J kg^{-1}, so escape needs K.E. per kg of 2.9×10^6 J; the rock has only $\frac{1}{2} \times 1800^2 = 1.6 \times 10^6$ J kg^{-1}. It cannot escape.