Mechanical Equilibrium

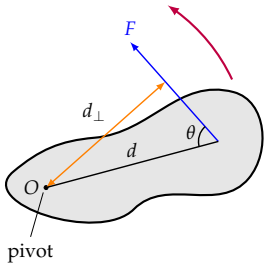
Name _____ Date _____

By the end of this topic I can...

- define the moment of a force, the torque of a couple and the centre of gravity
- state and use the principle of moments
- state the two conditions for equilibrium and apply them to beams, ladders and hanging objects
- use a closed force triangle for three forces in equilibrium

Moments and couples

For a point object zero resultant force is enough for equilibrium. For an extended (rigid) body a force can also have a turning effect.



Moment of a force about a pivot O.

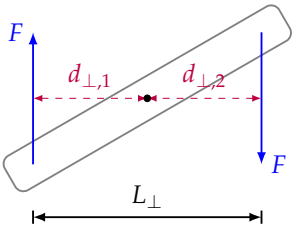
Moment of a force

The **moment** of a force about a point is the force multiplied by the perpendicular distance from the point to the line of action of the force:

$$\text{moment} = F \times d_\perp$$

[moment] = N m

- resolve either the force or the distance so that the two are perpendicular: a force F at angle θ to a lever of length d has moment $Fd\sin\theta$
- a moment is clockwise or anticlockwise
- its value depends on the pivot chosen: choose a pivot where an unknown force acts to remove it from the equation



A couple: two equal and opposite forces.

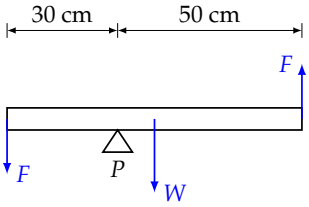
Torque of a couple

A **couple** is a pair of equal, opposite and parallel forces whose lines of action do not coincide. It produces a turning effect but no resultant force.

The **torque of a couple** is one of the forces multiplied by the perpendicular distance between the lines of action:

$$\tau = F\,L_\perp$$

The torque of a couple is the same about any point.



Example 4.1 The diagram shows a uniform beam of weight $W = 20$ N and length 80 cm pivoted at point P. P is 30 cm from one end. Two equal but opposite forces of magnitude $F = 12$ N are acting at the two ends of the beam as shown. What is the resultant moment about point P?

Moment of weight: $\tau_w = Wd_w = 20 \times \left(0.50 - \frac{1}{2} \times 0.80\right) = 2.0$ N m (clockwise)
Torque of couple: $\tau_c = FL = 12 \times 0.80 = 9.6$ N m (anti-clockwise)
Resultant moment: $\tau_{\text{net}} = \tau_c - \tau_W = 9.6 - 2.0 = 7.6$ N m (anti-clockwise)

Centre of gravity

The **centre of gravity** is the point where the whole weight of an object may be considered to act.

- a uniform regular object has its centre of gravity at its geometric centre
- a freely suspended object comes to rest with its centre of gravity vertically below the point of suspension
- to find it, suspend the object from two or more points and draw plumb lines; they cross at the centre of gravity

Stability

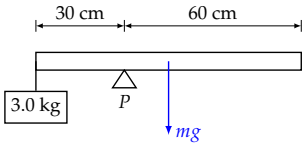
An object is **stable** if, when tilted slightly, the moment of its weight about the pivot edge turns it back. The weight must act on the inside of the base. Low centre of gravity and a wide base make an object more stable.

Equilibrium of rigid bodies

Principle of moments

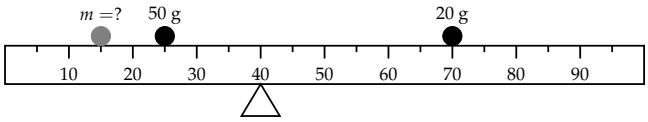
For a body in equilibrium the sum of the clockwise moments about any point equals the sum of the anticlockwise moments about that point:

$$\sum \tau_{\text{clockwise}} = \sum \tau_{\text{anticlockwise}}$$



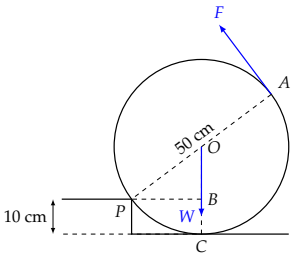
Example 4.2 A uniform rod of length 90 cm is pivoted 30 cm from one end. It is balanced with a 3.0 kg load. Find the mass of the rod.

Take moments about P:
$3.0 \times 9.81 \times 0.30 = m \times 9.81 \times (0.60 - \frac{1}{2} \times 0.90)$
so we find mass of rod: $m = 6.0$ kg



Example 4.3 A student balances a metre rule of mass 120 g supported on a fulcrum at the 40 cm mark. She then places a 20 g mass on the 70 cm mark and a 50 g mass on the 25 cm mark as shown. To balance the rule, what mass should she place on the 15 cm mark?

Moments about the support (the 40 cm mark), masses in grams, distances in cm:
anticlockwise: $m(40 - 15) + 50(40 - 25) = 25m + 750$
clockwise: $120(50 - 40) + 20(70 - 40) = 1200 + 600 = 1800$
so $25m = 1050$ and $m = 42$ g.



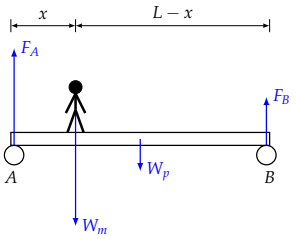
Example 4.4 A cylinder of weight 100 N and diameter 50 cm rests against point P of a curb of height 10 cm. What is the minimum force required to cause the cylinder to roll to the left?

To roll over the kerb the moment of F about P must at least equal the moment of the weight; F is smallest when its perpendicular distance from P is greatest ($|PA| = 50$ cm, the diameter).
Horizontal distance of the weight from P: $|PB| = \sqrt{0.25^2 - 0.15^2} = 0.20$ m.
$F_{\min} \times 0.50 = 100 \times 0.20 \quad \Rightarrow \quad F_{\min} = 40$ N

The conditions for equilibrium

A body is in equilibrium when

- the resultant force is zero : $\sum F = 0$ in any direction
- the resultant moment about any point is zero



Example 4.5 A uniform plank of weight 100 N and length $L = 6.0$ m rests horizontally on two supports A and B. A man of weight 800 N stands a distance of $x = 1.5$ m from end A. Determine the forces acting at the two supports.

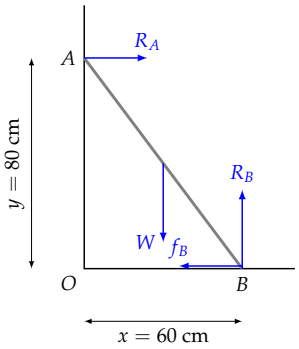
Take moments about A: $W_m x + W_p \cdot \frac{L}{2} = F_B L$

$$800 \times 1.5 + 100 \times 3.0 = F_B \times 6.0 \Rightarrow F_B = 250 \text{ N}$$

Take moments about B: $W_m(L - x) + W_p \frac{L}{2} = F_A L$

$$800 \times 4.5 + 100 \times 3.0 = F_A \times 6.0 \Rightarrow F_B = 650 \text{ N}$$

One can check that: $F_A + F_B = W_m + W_p$, there must be no resultant force in vertical direction



Example 4.6 A uniform ladder of weight 120 N rests on a rough ground against a smooth wall as shown. The dimensions are labelled on the diagram. (a) What is the contact force acting at B? (b) What is the contact force acting at A? (c) What is the frictional force at B?

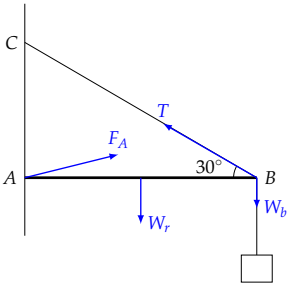
Free-body diagram is drawn as shown

resolve vertically: $R_B = W \quad \Rightarrow \quad R_B = 120 \text{ N}$

take moments about B: $R_A y = W \frac{x}{2}$

$$R_A = \frac{120 \times 0.30}{0.80} = 45 \text{ N}$$

resolve horizontally: $f_B = R_A \quad \Rightarrow \quad f_B = 45 \text{ N}$



Example 4.7 The diagram shows a uniform rod AB of weight 60 N being held horizontally to a vertical wall by means of a light string. The string is attached to the rod at B, where a basket of weight 40 N is suspended. The other end of the string is fixed on the wall at C. The angle between the string and the rod is 30°. (a) Find the tension in the string. (b) Find the force acting on the rod at point A.

Take moments about A: $TL\sin\theta = W_bL + W_r\frac{1}{2}L$

$$T\sin 30^\circ = 40 + 60\times\tfrac{1}{2} \quad\Rightarrow\quad T = 140\text{ N}$$

resolve horizontally: $F_{A,x} = T\cos\theta \quad\Rightarrow\quad F_{A_x} = 140\cos 30^\circ \approx 121\text{ N}$

resolve vertically: $F_{A,y} + T\sin\theta = W_r + W_b \quad\Rightarrow\quad F_{A,y} = 60 + 40 - 140\sin 30^\circ = 30\text{ N}$

force at A: $F_A = \sqrt{F_{A,x}^2 + F_{A,y}^2} \quad\Rightarrow\quad F_A = \sqrt{121^2 + 30^2} \approx 125\text{ N}$

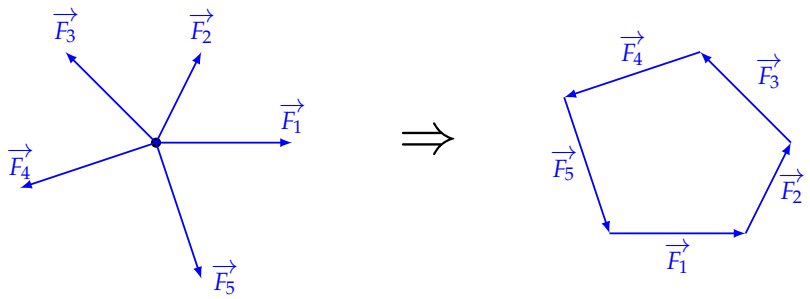
Forces in equilibrium

Two forces

Two forces in equilibrium must be ___equal and opposite___ (zero resultant force) and act along the ___same line___ (otherwise they form a couple).

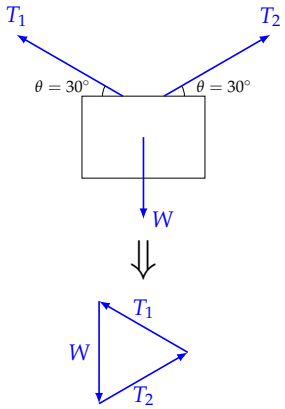
Three forces

- zero resultant force: the three force vectors form a closed **force triangle** when joined ___head to tail___
- zero resultant moment: the three lines of action must pass through ___one point___ (they are **concurrent**)



A set of forces in equilibrium forms a closed polygon.

Example 4.8 A painting of weight $W = 20$ N is supported by two strings as shown. Both strings form an angle $\theta = 30°$ to the horizontal. Find the tension in the strings.



By resolving forces, we have:

$$\begin{cases} T_1 \cos\theta = T_2 \cos\theta \\ T_1 \sin\theta + T_2 \sin\theta = W \end{cases}$$

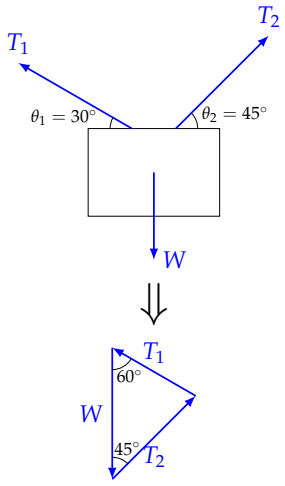
we solve the equations to obtain:

$$T_1 = T_2 = \frac{W}{2\sin\theta} = \frac{20}{2\sin 30°} = 20 \text{ N}$$

alternatively, we can construct the force triangle as shown
T_1, T_2 and W form an equilateral triangle, so

$$T_1 = T_2 = W = 20 \text{ N}$$

Example 4.9 The same painting of weight $W = 20$ N is supported by two strings at different angles $\theta_1 = 30°$ and $\theta_2 = 45°$ as shown. Find the forces in the two strings.



By resolving forces, we have:

$$\begin{cases} T_1 \cos\theta_1 = T_2 \cos\theta_2 \\ T_1 \sin\theta_1 + T_2 \sin\theta_2 = W \end{cases} \Rightarrow \begin{cases} \frac{\sqrt{3}}{2} T_1 = \frac{\sqrt{2}}{2} T_2 \\ \frac{1}{2} T_1 + \frac{\sqrt{2}}{2} T_2 = 20 \end{cases}$$

solving this, we find: $T_1 \approx 14.6$ N, $T_2 \approx 17.9$ N
alternatively, we construct the force triangle as shown
T_1 and T_2 are related to W by the the law of sine:

$$\frac{W}{\sin 75°} = \frac{T_1}{\sin 45°} = \frac{T_2}{\sin 60°}$$

from this we get the same result: $T_1 \approx 14.6$ N, $T_2 \approx 17.9$ N

Practice questions

Question 4.1 Is it possible for an object to be in equilibrium if only one force is acting on it?

No. One force alone is a non-zero resultant force, so by Newton's first law the object cannot be in equilibrium (the other forces would need to cancel it).

Question 4.2 If three forces are in equilibrium, suggest and explain whether the lines of action must lie in the same plane.

Yes. Three forces in equilibrium form a closed triangle, and a triangle lies in one plane, so the forces are coplanar. A force out of the plane could not be cancelled by the other two.

Question 4.3 Two children sit on a uniform see-saw balanced on its centre. A child of mass 25 kg sits 1.8 m from the pivot. Where must a child of mass 30 kg sit to balance it?

Moments about the pivot: $25g \times 1.8 = 30g \times d \quad \Rightarrow \quad d = \dfrac{25 \times 1.8}{30} = 1.5$ m on the other side.

Question 4.4 A driver turns a steering wheel of diameter 0.40 m by applying a force of 15 N with each hand, in opposite directions on opposite sides of the wheel. What is the torque applied?

This is a couple: $\tau = FL_\perp = 15 \times 0.40 = 6.0$ N m.