Electrostatics

Name _____ Date _____

By the end of this topic I can...

- use Coulomb's law, and define and use electric field strength E
- draw electric field lines and equipotentials
- use electric potential energy and potential: $E_p = \frac{Qq}{4\pi\epsilon_0 r}$, $V = \frac{Q}{4\pi\epsilon_0 r}$
- use $E = -\mathrm{d}V/\mathrm{d}r$, $E = V/d$, and compare electric and gravitational fields

Electric forces and fields

Coulomb's law

Charles Augustin de Coulomb published this law in 1785. It tells us how charged objects attract or repel through the _____electric force_____.

_____Coulomb's law_____: the electric force between two point charges is _____proportional to the product of their charges_____ and _____inversely proportional to the square of their separation_____:

$$F = \frac{Qq}{4\pi\epsilon_0 r^2}$$

$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$ is the _____permittivity of free space_____, and $k = \dfrac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\ \mathrm{N\,m^2\,C^{-2}}$.

- Q and q are _____point charges_____; a uniformly charged sphere acts as a point charge, with r measured between _____centres_____
- like charges: $Q_1Q_2 > 0$, so $F > 0$: _____repulsion_____; opposite charges: $Q_1Q_2 < 0$, so $F < 0$: _____attraction_____

Example 23.1 A hydrogen atom has radius about 53 pm. Estimate the electric force between the proton and the electron.

$$F = \frac{Q_pQ_e}{4\pi\epsilon_0 r^2} = 8.99 \times 10^9 \times \frac{(1.60 \times 10^{-19})^2}{(53 \times 10^{-12})^2} \approx 8.2 \times 10^{-8}\ \mathrm{N}$$

Question 23.1 Two protons are a distance r apart. Find the ratio of the electric force to the gravitational force between them.

$$\frac{F_E}{F_G} = \frac{e^2/4\pi\epsilon_0 r^2}{Gm_p^2/r^2} = \frac{8.99 \times 10^9 \times (1.60 \times 10^{-19})^2}{6.67 \times 10^{-11} \times (1.67 \times 10^{-27})^2} \approx 1.2 \times 10^{36}$$

independent of r.

Electric fields

An electric field is a region of space where a charged object experiences a force.

Coulomb's law and Newton's law of gravitation are both inverse square laws, so electric and gravitational fields are very similar.

Electric field strength

Electric field strength is the electric force per unit positive charge:

$$E = \frac{F}{q}$$

- unit: $[E] =$ $\mathrm{N\,C^{-1}}$ $= \mathrm{V\,m^{-1}}$
- for a point charge Q, a test charge q at distance r feels $F = \dfrac{Qq}{4\pi\epsilon_0 r^2}$, so

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

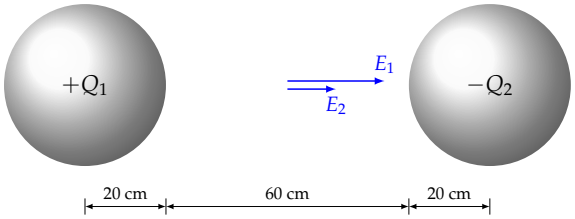
The field depends on the source Q only. A charged sphere (outside, $r > R$) acts like a point charge at its centre.

- E is a vector: it points away from a positive source and towards a negative one. For several charges, add as vectors.
- the force on a charge q is $F = Eq$: in the direction of E if $q > 0$, opposite to E if $q < 0$

Example 23.2 A Van de Graaff generator sparks when the surface field is 4.0×10^4 V cm^{-1}. Its sphere has diameter 40 cm. What is the charge on it?

$E = 4.0 \times 10^6$ V m^{-1}, $r = 0.20$ m:

$$Q = 4\pi\epsilon_0 E r^2 = 4\pi \times 8.85 \times 10^{-12} \times 4.0 \times 10^6 \times 0.20^2 \approx 1.8 \times 10^{-5}\ \text{C}$$



Example 23.3 Two identical metal spheres of radius 20 cm carry charges $+2.0\ \mu$C and $-1.0\ \mu$C, with a 60 cm gap between them. (a) Find the field strength midway between their centres. (b) A dust particle with charge -1.3×10^{-8} C is placed there. Find the force on it.

The midpoint is $20 + 30 = 50$ cm $= 0.50$ m from each centre. The fields point the same way (away from $+$, towards $-$):

$$E = E_1 + E_2 = 8.99 \times 10^9 \left(\frac{2.0 \times 10^{-6}}{0.50^2} + \frac{1.0 \times 10^{-6}}{0.50^2} \right) \approx 1.08 \times 10^5\ \text{N C}^{-1}$$

towards the negative sphere (to the right). (b) $F = Eq = 1.08 \times 10^5 \times 1.3 \times 10^{-8} \approx 1.4 \times 10^{-3}$ N; the charge is negative, so the force is opposite to E (to the left).

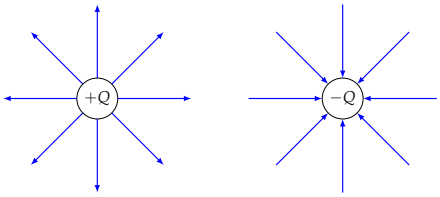
Question 23.2 At charge 4.0×10^{-7} C the field at the surface of a Van de Graaff sphere is 2.4×10^6 V m^{-1}. Find the extra charge needed to raise the surface field to 3.0×10^6 V m^{-1}.

$E \propto Q$ for a fixed sphere, so $Q' = 4.0 \times 10^{-7} \times \dfrac{3.0}{2.4} = 5.0 \times 10^{-7}$ C. Extra charge $= 1.0 \times 10^{-7}$ C.

Question 23.3 Two positively charged particles A and B are in a vacuum. P is on the line joining them, a distance x from A. Sketch how the field strength E varies with x.

Taking the direction from A to B as positive: $E \to +\infty$ as $x \to 0$ (near A, pushed towards B), falling to zero at the point between them where the two fields cancel, then negative, and $\to -\infty$ as P approaches B.

Electric field lines

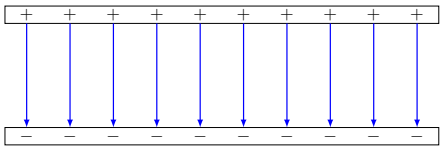


Field of a positive and a negative sphere.

Field lines show direction (the arrows) and strength (the spacing). They leave positive charges and end on negative charges.

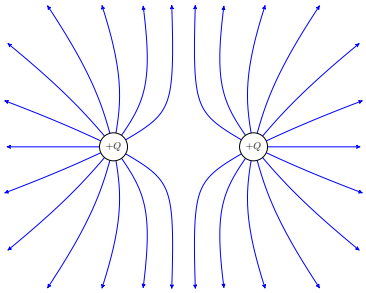
Example 23.4 Sketch the field of a positive or a negative charged sphere, and explain why they act as point charges.

The lines are radial: perpendicular to the surface, appearing to start from (or converge on) the centre. So viewed from outside, a charged sphere behaves like a point charge at its centre.

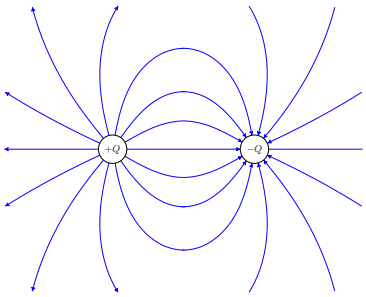

Field between oppositely charged plates.

Example 23.5 Sketch the field between two large oppositely charged metal plates.

The lines are parallel and equally spaced: a uniform field.



Two equal positive charges.



Two opposite charges.

Electric potential and potential energy

Electric potential energy

As for gravity, the change in electric P.E. is the work done ___against___ (or by) the electric force, with $E_p = 0$ at $r = \infty$.

___Electric potential energy___ is the ___work done by the electric force___ bringing a charge to a position from ___infinity___.

Bringing q from infinity to r from Q: $W = \int_{\infty}^{r} F\,\mathrm{d}x = -\dfrac{Qq}{4\pi\epsilon_0 r}$, and $\Delta E_p = -W$, so

$$E_p = \frac{Qq}{4\pi\epsilon_0 r}$$

• like charges: $Qq > 0$, $E_p > 0$; ___work must be done___ to push them together (repulsion)

• opposite charges: $Qq < 0$, $E_p < 0$ (attraction); pulling them apart needs work, so E_p ___increases___ towards zero

• E_p is a ___scalar___, but its sign matters: repulsion means positive, attraction means ___negative___

Example 23.6 Alpha particles ($^4_2\alpha$) are fired at a gold foil ($^{197}_{79}$Au). A gold nucleus has radius about 10^{-14} m. What minimum initial speed lets an α-particle get this close?

The α-particle slows as it approaches (repulsion): K.E. is converted to electric P.E. At closest approach $v = 0$:

$$\tfrac{1}{2}mu^2 = \frac{Qq}{4\pi\epsilon_0 r}$$

$$u^2 = \frac{2}{4 \times 1.66 \times 10^{-27}} \times \frac{79 \times 2 \times (1.60 \times 10^{-19})^2}{4\pi \times 8.85 \times 10^{-12} \times 10^{-14}}$$

$$u \approx 3.3 \times 10^7 \text{ m s}^{-1}$$

Question 23.4 A metal sphere of radius 20 cm carries 5.0×10^{-7} C. A proton is fired at it at 1.8×10^6 m s^{-1}. Can it reach the surface?

K.E. $= \frac{1}{2} \times 1.67 \times 10^{-27} \times (1.8 \times 10^6)^2 = 2.7 \times 10^{-15}$ J.
P.E. gain to reach the surface:

$$\frac{Qe}{4\pi\epsilon_0 R} = \frac{8.99 \times 10^9 \times 5.0 \times 10^{-7} \times 1.60 \times 10^{-19}}{0.20} = 3.6 \times 10^{-15} \text{ J}$$

This is more than the K.E., so the proton **cannot** reach the surface.

Electric potential

Electric potential at a point is the work done bringing a unit positive charge from infinity to that point: $V = \frac{E_p}{q}$

- unit: $[V] = \text{J C}^{-1} =$ V
- for a point charge Q:

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

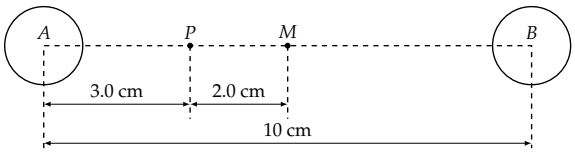
- $V =$ 0 at infinity; $V > 0$ near a positive source, $V < 0$ near a negative one

- V is a scalar: add the contributions from several charges as scalars

Example 23.7 An electron is accelerated from rest through a p.d. of 600 V. Find its final speed.

Gain in K.E. = loss in electric P.E.: $\frac{1}{2}mv^2 = q\Delta V$

$$v = \sqrt{\frac{2q\Delta V}{m}} = \sqrt{\frac{2 \times 1.60 \times 10^{-19} \times 600}{9.11 \times 10^{-31}}} \approx 1.45 \times 10^7 \text{ m s}^{-1}$$



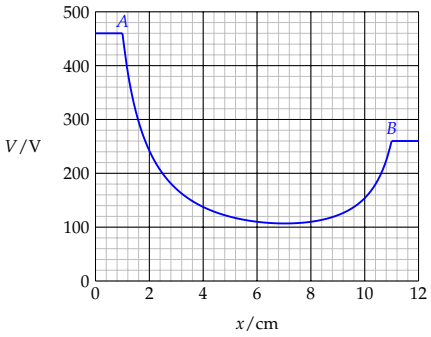
Example 23.8 Spheres A (+20 pC) and B (+84 pC) are 10 cm apart in a vacuum. P is 3.0 cm from A and M is the midpoint. (a) Find the potential at P and at M. (b) Find the work needed to move an α-particle from P to M.

$$V = V_A + V_B = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_A}{r_A} + \frac{Q_B}{r_B}\right)$$

$$V_P = 8.99 \times 10^9 \left(\frac{20 \times 10^{-12}}{0.030} + \frac{84 \times 10^{-12}}{0.070}\right) \approx 16.8 \text{ V}$$

$$V_M = 8.99 \times 10^9 \left(\frac{20 \times 10^{-12}}{0.050} + \frac{84 \times 10^{-12}}{0.050}\right) \approx 18.7 \text{ V}$$

$$W = q\Delta V = 2 \times 1.60 \times 10^{-19} \times (18.7 - 16.8) \approx 6.2 \times 10^{-19} \text{ J}$$



Example 23.9 Spheres A and B (radius 1.0 cm) are positively charged. A proton starts at rest on the surface of A and moves along the line to B. The graph shows the potential V against distance x from the centre of A. (a) Find its maximum speed. (b) Find its speed on reaching the surface of B.

Increase in K.E. = loss in P.E.: $\frac{1}{2}mv^2 = q(V_A - V_P)$

(a) Fastest where V_P is smallest (107 V at $x = 7.2$ cm):

$$\tfrac{1}{2} \times 1.67 \times 10^{-27} v_{\max}^2 = 1.60 \times 10^{-19} \times (460 - 107) \quad \Rightarrow \quad v_{\max} \approx 2.60 \times 10^5 \text{ m s}^{-1}$$

(b) At the surface of B, $V = 260$ V:

$$\tfrac{1}{2} \times 1.67 \times 10^{-27} v_B^2 = 1.60 \times 10^{-19} \times (460 - 260) \quad \Rightarrow \quad v_B \approx 1.96 \times 10^5 \text{ m s}^{-1}$$

Question 23.5 Breakdown occurs when the surface field of a metal sphere of radius 16 cm exceeds 5.0×10^6 N C^{-1}. What is the surface potential at breakdown?

$$V = Er = 5.0 \times 10^6 \times 0.16 = 8.0 \times 10^5 \text{ V}$$

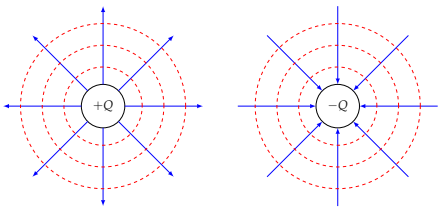
Question 23.6 A and B are 20 cm apart; A has charge $+7.2\ \mu$C. The potential is zero at the point P on AB with $AP = 5.0$ cm (between them). Find the charge of B.

$$V_P = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_A}{0.050} + \frac{Q_B}{0.150}\right) = 0 \quad \Rightarrow \quad Q_B = -3Q_A = -21.6\ \mu\text{C}$$

Question 23.7 A particle of specific charge $+9.58 \times 10^7$ C kg^{-1} approaches a fixed metal sphere at potential +500 V, with initial speed 3.0×10^5 m s^{-1} from far away. Can it reach the surface?

Per kilogram: K.E. $= \frac{1}{2}v^2 = 4.5 \times 10^{10}$ J kg^{-1}.
P.E. to gain: $\frac{q}{m}V = 9.58 \times 10^7 \times 500 = 4.8 \times 10^{10}$ J kg^{-1}.
This exceeds the K.E., so it **cannot** reach the surface.

Equipotentials



Field lines (blue) and equipotentials (red dashed).

Equipotential lines join points of the same potential; they are the contour lines of potential.

- point charge: equipotentials are concentric circles
- uniform field: equipotentials are parallel straight lines
- equipotentials are always perpendicular to field lines, and moving along one needs no work

Further ideas

Comparison with gravitational fields

	gravitational	electric
force	$F = -\frac{GMm}{r^2}$	$F = \frac{Qq}{4\pi\epsilon_0 r^2}$
field strength	$g = -\frac{GM}{r^2}$	$E = \frac{Q}{4\pi\epsilon_0 r^2}$
potential energy	$E_p = -\frac{GMm}{r}$	$E_p = \frac{Qq}{4\pi\epsilon_0 r}$
potential	$\varphi = -\frac{GM}{r}$	$V = \frac{Q}{4\pi\epsilon_0 r}$

- *similar*: both are inverse square laws; E_p and potential vary as $1/r$; both are zero at infinity
- *different*: mass is always positive but charge can be either sign, so electric force can be repulsive or attractive, gravity is always attractive; electric potential takes both signs, gravitational potential is always negative

Fields inside conductors

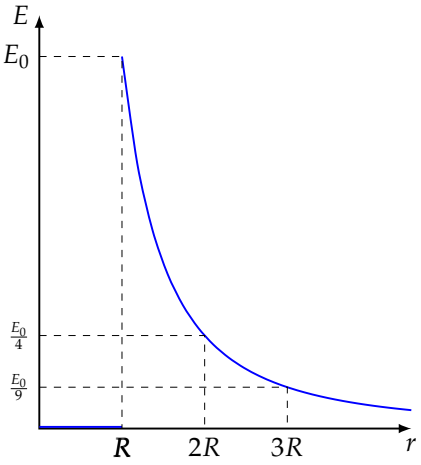
A charged conductor has free charges. They settle so that no net force acts on them.

Inside a conductor: field strength $E =$ 0 everywhere, and potential $V =$ constant everywhere.

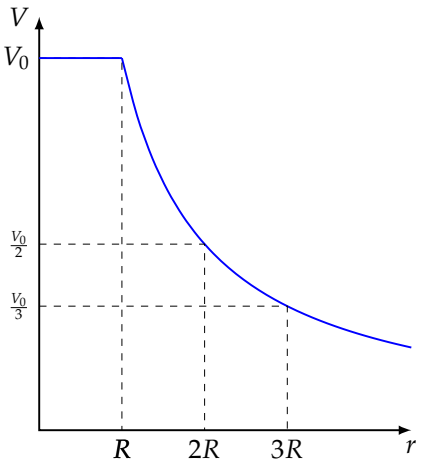
Example 23.10 Plot E and V against distance r from the centre of a metal sphere of radius R carrying charge Q.

Outside ($r > R$) the sphere acts as a point charge: $E = \dfrac{Q}{4\pi\epsilon_0 r^2}$, $V = \dfrac{Q}{4\pi\epsilon_0 r}$.

Inside ($r < R$): $E = 0$ and $V = \dfrac{Q}{4\pi\epsilon_0 R} =$ constant.



E against r for a charged metal sphere.



V against r for the same sphere.

Question 23.8 Are the two spheres in the earlier V–x graph example conductors? What feature of the graph supports this?

Yes: inside each sphere the potential is **constant** (flat sections on the graph), as it must be inside a conductor.

Question 23.9 Why does a mobile phone signal weaken inside a lift?

The metal lift acts as a conducting shell. Free charges rearrange to cancel the field inside, so the radio waves are screened out (a Faraday cage).

Field strength and potential

For a small displacement Δr: $\Delta V = \dfrac{\Delta E_p}{q} = -\dfrac{F\Delta r}{q} = -E\,\Delta r$, so

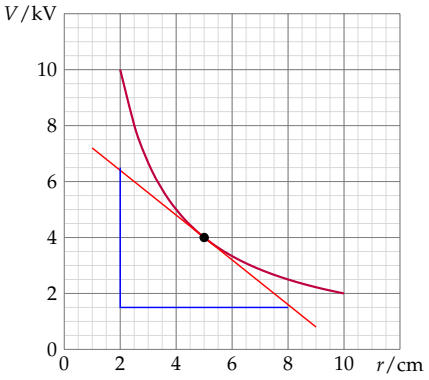
$$E = -\frac{\mathrm{d}V}{\mathrm{d}r}$$

Field strength is the negative gradient of the potential with respect to displacement.

Conversely

$$\Delta V = -\int E \, \mathrm{d}r$$

- on a V–r graph the gradient gives E; on an E–r graph the area gives ΔV
- similarly $F = -\mathrm{d}E_p/\mathrm{d}r$: the force acts to lower the potential energy



V (kV) against r (cm) near a charged object.

Example 23.11 The graph shows the potential near a charged object, with the tangent at $r = 5.0$ cm drawn. Find the field strength at 5.0 cm.

Gradient of the tangent (from (2.0 cm, 6.4 kV) to (8.0 cm, 1.6 kV)):

$$\frac{\Delta V}{\Delta r} = \frac{(1.6 - 6.4) \times 10^3}{(8.0 - 2.0) \times 10^{-2}} = -8.0 \times 10^4 \text{ V m}^{-1}$$

$$E = -\frac{\Delta V}{\Delta r} = 8.0 \times 10^4 \text{ V m}^{-1}$$

Question 23.10 Show that the object in the previous example behaves as a point charge, find its charge, and hence E at 5.0 cm.

$V = 20 \text{ kV cm}/r$, so $Vr = 20 \times 10^3 \times 0.01 = 200$ V m is constant, as for $V = \dfrac{Q}{4\pi\epsilon_0 r}$.

$$Q = 4\pi\epsilon_0 Vr = \frac{200}{8.99 \times 10^9} \approx 2.2 \times 10^{-8} \text{ C}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{8.99 \times 10^9 \times 2.2 \times 10^{-8}}{0.050^2} \approx 8.0 \times 10^4 \text{ V m}^{-1}$$

Example 23.12 Differentiate $V = \dfrac{Q}{4\pi\epsilon_0 r}$ to find E for a point charge.

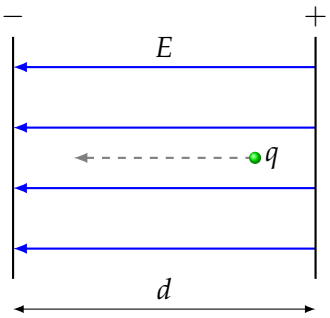
$$E = -\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{Q}{4\pi\epsilon_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right) = \frac{Q}{4\pi\epsilon_0 r^2}$$

This agrees with the field strength found earlier.

Question 23.11 Does 'field strength equals minus the potential gradient' also hold for gravitational fields?

Yes: $g = -\frac{\mathrm{d}\varphi}{\mathrm{d}r}$. With $\varphi = -\frac{GM}{r}$, $\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{GM}{r^2}$, so $g = -\frac{GM}{r^2}$, pointing towards the mass.

Uniform fields



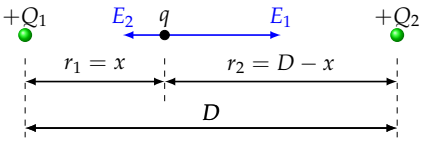
Uniform field between plates, and the linear V–r graph.

For plates a distance d apart with p.d. V: moving q through the field, $W = Fd = Eqd$, so $\Delta E_p = -Eqd$ and $\Delta V = -Ed$ in magnitude:

$$E = \frac{V}{d}$$

The V–r graph is a straight line with gradient $-E$. The field points from ___high___ to ___low___ potential.

Two positive point charges

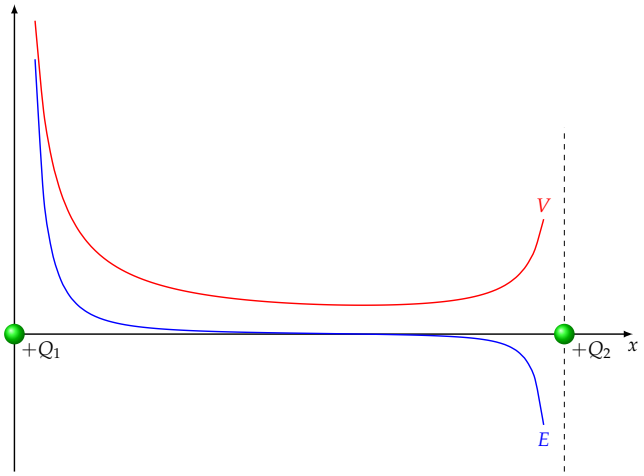


Charges $+Q_1$ and $+Q_2$, distance D apart.

Along the line joining them, at distance x from Q_1:

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{x} + \frac{Q_2}{D-x}\right) \qquad E = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{x^2} - \frac{Q_2}{(D-x)^2}\right)$$

V is a ___scalar sum___, E a ___vector sum___ (the two fields point in opposite directions between the charges).



Question 23.12 Discuss the field between (a) two negative charges, (b) two opposite charges.

(a) The pattern is the same as for two positive charges with all field directions reversed: $E = 0$ at a point between them, and $V < 0$ everywhere, rising to zero at infinity.
(b) The fields from the two charges point the same way between them, so E never reaches zero there; V passes through zero (at the midpoint if the charges are equal).