Circular Motion

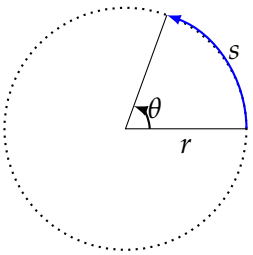
Name _____ Date _____

By the end of this topic I can...

- define angular displacement and the radian, and use $s = r\theta$
- define angular velocity and use $\omega = \Delta\theta/\Delta t$, $v = \omega r$ and $\omega = 2\pi/T$
- explain why uniform circular motion is accelerated and use $a = v^2/r = \omega^2 r$
- identify what provides the centripetal force and use $F = mv^2/r = m\omega^2 r$

Angular quantities

Movement of an object along a circular path is called circular motion. To describe it we use angular quantities, which turn out to be more useful than linear displacement, linear velocity, etc.



Angular displacement

Angular displacement is the angle swept out by an object moving along a circular path.

- unit: $[\theta] =$ radian (rad) (the natural unit for angles)
- conversion: 2π rad $=$ 360°
- if two radii form an angle θ, the length of arc between them is

$$s = r\theta$$

- one **radian** is the angle formed when an arc has the same length as the radius.

Angular velocity

Angular velocity describes how fast an object moves along a circular path.

Angular velocity is the angular displacement swept out per unit time:

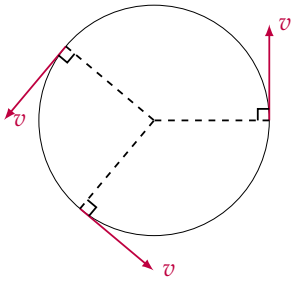
$$\omega = \frac{\Delta\theta}{\Delta t}$$

- unit: $[\omega] =$ rad s^{-1}
- angular velocity is really a *vector* (normal to the plane of the motion) but at A-level we treat it as a scalar, so angular velocity and angular speed are the same idea.

Angular and linear velocity are closely related. In time Δt the distance moved along the arc is

$$\Delta s = v\,\Delta t = r\,\Delta\theta \quad \Rightarrow \quad \omega = \frac{\Delta\theta}{\Delta t} = \frac{v}{r} \quad \Rightarrow \quad \boxed{v = \omega r}$$

This holds at any instant.



Uniform circular motion

The simplest circular motion has constant ω (constant speed). Compare it with uniform linear motion, $s = vt$: displacement $s \leftrightarrow$ θ, velocity $v \leftrightarrow$ ω. So

$$\boxed{\theta = \omega t}$$

- the time for one complete revolution is the period, T.

In one period the angle swept out is 2π, so

$$\boxed{\omega = \frac{2\pi}{T}}$$

- uniform circular motion is still *accelerated* motion: the speed is constant but the velocity is changing, because its direction is always tangential to the path and so keeps changing.

Example 18.1 An object moves at constant speed around a circular track of radius 2.5 m, taking 40 s for one lap. What are its angular speed and linear speed?

$\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{40} \approx 0.157$ rad s^{-1} $\qquad v = \omega r = 0.157 \times 2.5 \approx 0.39$ m s^{-1}

Question 18.1 What is the angular velocity of the minute hand of a clock?

$T = 3600$ s, so $\omega = \dfrac{2\pi}{3600} \approx 1.7 \times 10^{-3}$ rad s^{-1}

Question 18.2 A spacecraft moves around the Earth in a circular orbit at a speed of 7200 m s^{-1}, at a height of 1300 km above the surface. The radius of the Earth is 6400 km. (a) What is the angular speed of the spacecraft? (b) What is its period?

$r = 6400 + 1300 = 7700$ km $= 7.7 \times 10^6$ m

(a) $\omega = \dfrac{v}{r} = \dfrac{7200}{7.7 \times 10^6} \approx 9.4 \times 10^{-4}$ rad s^{-1}

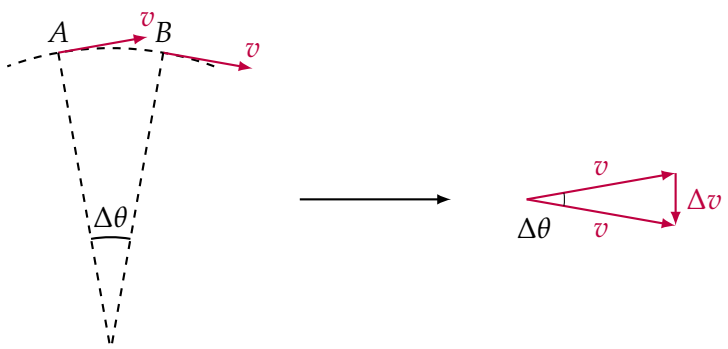
(b) $T = \dfrac{2\pi}{\omega} \approx 6.7 \times 10^3$ s (≈ 1.9 h)

Centripetal acceleration

Centripetal acceleration is the acceleration due to the change in direction of the velocity vector. It points towards the centre of the circular path.

Consider motion from A to B at constant speed v, in a very short time Δt.[1]

[1] A more rigorous derivation uses differentiation.



change in velocity (small angle, $\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$): $\Delta v = 2v \sin\frac{\Delta\theta}{2} \approx$ $v\,\Delta\theta$

acceleration: $a = \dfrac{\Delta v}{\Delta t} \approx v\dfrac{\Delta\theta}{\Delta t} =$ $v\omega$

using $v = \omega r$:

$$a_c = \frac{v^2}{r} = \omega^2 r$$

- the direction of a_c is always towards the centre of the circle.
- centripetal acceleration only changes the *direction* of the velocity. A change in the *magnitude* of the velocity needs a tangential acceleration, which is linked to angular acceleration (*beyond the syllabus*).[2]

[2] Angular acceleration, $\alpha = \mathrm{d}\omega/\mathrm{d}t$, is the rate of change of angular velocity and is analogous to linear acceleration a. Just as $v = \omega r$, we also have $a = \alpha r$.

Question 18.3 A racing car makes a 180° turn in 2.0 s. Assume the path is a semicircle of radius 30 m and the car keeps a constant speed. (a) What is the angular velocity of the car? (b) What is its centripetal acceleration?

(a) $180^\circ = \pi$ rad, so $\omega = \dfrac{\pi}{2.0} \approx 1.6$ rad s^{-1}
(b) $a_c = \omega^2 r = 1.57^2 \times 30 \approx 74$ m s^{-2}

Centripetal force

Circular motion involves a changing velocity, so the object is not in equilibrium. There must be a net (resultant) force on it.

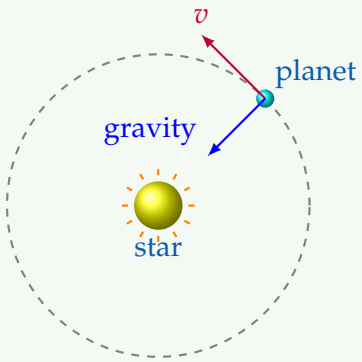
Centripetal force F_c is the resultant force acting on an object moving along a circular path, and it is always directed towards the centre of the circle.

- centripetal force causes centripetal acceleration. From Newton's second law:

$$F_c = \frac{mv^2}{r} = m\omega^2 r$$

- F_c is not a new kind of force. It is the resultant of forces you already know: weight, tension, contact force, friction, ...
- F_c acts at right angles to the velocity. If $F_{\text{net}} \perp v$ and F_{net} has constant magnitude, the object moves in a circle.
- the effect of F_c is to change the *direction* of motion (or to hold a circular orbit). Changing the *magnitude* of the velocity needs a tangential component of force (beyond the syllabus).

Example 18.2 A planet orbits a star. Sketch the orbit, mark the planet's velocity and name the force that provides the centripetal force.



Gravity from the star provides the centripetal force on the planet.

Example 18.3 A rock orbits the Earth just above the surface. Ignore air resistance, so only weight acts. The Earth's radius is $R = 6400$ km. (a) What is the orbital speed? (b) What is the orbital period?

Weight provides the centripetal force: $mg = \dfrac{mv^2}{R}$

(a) $v = \sqrt{gR} = \sqrt{9.81 \times 6.4 \times 10^6} \approx 7.9 \times 10^3$ m s^{-1}

(b) $T = \dfrac{2\pi R}{v} = \dfrac{2\pi \times 6.4 \times 10^6}{7.9 \times 10^3} \approx 5.1 \times 10^3$ s ≈ 85 min

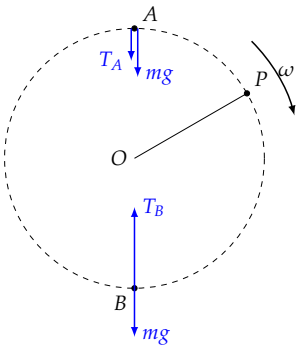
Example 18.4 A turntable rotates freely about a vertical axis through its centre. A small object sits on it at $d = 40$ cm from the centre. The angular speed is slowly increased. The coefficient of friction between object and turntable is $\mu = 0.30$. Find the maximum number of revolutions per minute before the object slides.

Friction provides the centripetal force: $f = m\omega^2 d$. Increasing ω needs more friction, but $f_{\text{lim}} = \mu N = \mu mg$, so

$$m\omega^2 d \leq \mu mg \quad \Rightarrow \quad \omega^2 \leq \frac{\mu g}{d}$$

$$\omega_{\text{max}} = \sqrt{\frac{0.30 \times 9.81}{0.40}} \approx 2.71 \text{ rad s}^{-1}$$

$$T_{\text{min}} = \frac{2\pi}{\omega_{\text{max}}} \approx 2.32 \text{ s} \quad \text{so} \quad n_{\text{max}} = \frac{60}{2.32} \approx 25.9 \text{ revolutions per minute}$$

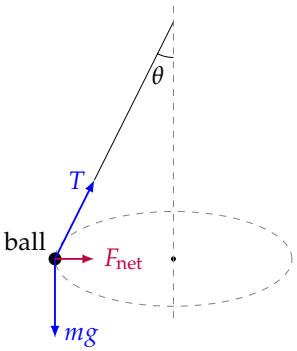


Example 18.5 A particle P of mass $m = 0.40$ kg is on a light inextensible string of length $r = 0.80$ m and is whirled at constant angular speed ω in a vertical plane. (a) If the string never goes slack, find the minimum ω. (b) If the string breaks when the tension exceeds 20 N, find the maximum ω.

Top (A): $F_c = T_A + mg = m\omega^2 r \quad \Rightarrow \quad T_A = m\omega^2 r - mg$
Bottom (B): $F_c = T_B - mg = m\omega^2 r \quad \Rightarrow \quad T_B = m\omega^2 r + mg$

(a) Tension is least at A; taut means $T_A \geq 0$: $\omega_{\min} = \sqrt{\dfrac{g}{r}} = \sqrt{\dfrac{9.81}{0.80}} \approx 3.5$ rad s^{-1}

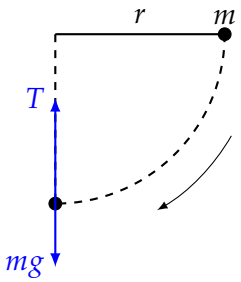
(b) Tension is greatest at B; $T_B \leq T_{\max}$: $\omega_{\max} = \sqrt{\dfrac{T_{\max}}{m} - \dfrac{g}{r}} = \sqrt{\dfrac{20}{0.40} - \dfrac{9.81}{0.80}} \approx 6.1$ rad s^{-1}



Example 18.6 A pendulum bob of mass 120 g moves at constant speed in a horizontal circle of radius $r = 10$ cm. The string makes an angle $\theta = 25°$ with the vertical. (a) What is the tension? (b) How fast is the bob moving?

(a) Vertical: $T\cos\theta = mg \quad \Rightarrow \quad T = \dfrac{0.12 \times 9.81}{\cos 25°} \approx 1.3$ N

(b) Horizontal component of T provides F_c: $T\sin\theta = \dfrac{mv^2}{r}$. Eliminating T and m: $v^2 = rg\tan\theta = 0.10 \times 9.81 \times \tan 25° \quad \Rightarrow \quad v \approx 0.68$ m s^{-1}



Example 18.7 A small ball of mass m is on an inextensible string of length l, held taut and horizontal, then released from rest. At the lowest point, find its speed and the tension in the string, in terms of m and l.

Energy conservation (GPE lost = KE gained): $mgl = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{2gl}$

Lowest point: $F_c = T - mg = \dfrac{mv^2}{l} \quad \Rightarrow \quad T = mg + \dfrac{m(2gl)}{l} = 3mg$

Question 18.4 Suggest what provides the centripetal force in each case. (a) An athlete running on a curved track. (b) An aeroplane banking at constant altitude. (c) A satellite orbiting the Earth.

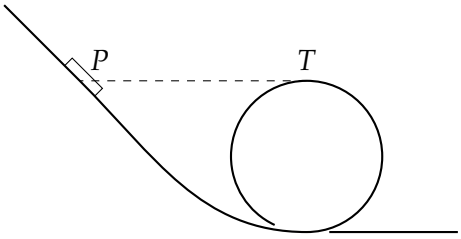
(a) friction between shoes and track (b) horizontal component of the lift on the wings (c) gravity (the satellite's weight)

Question 18.5 A turntable that rotates freely in a horizontal plane is covered in dry mud. The angular speed is gradually increased. State and explain whether the mud near the edge or near the centre leaves first.

Near the edge. The required $F_c = m\omega^2 r$ grows with r, while the maximum friction/adhesion is the same, so the limit is reached first at the largest r.

Question 18.6 A bucket of water is swung at constant speed in a vertical circle of radius $r = 1.0$ m. If the water stays in the bucket at the highest point, how fast must the bucket be swung?

At the top $N + mg = \dfrac{mv^2}{r}$; water stays in if $N \geq 0$, so $v \geq \sqrt{gr} = \sqrt{9.81 \times 1.0} \approx 3.1\ \text{m s}^{-1}$



Question 18.7 Design a roller-coaster. A slider starts from rest at P and slides along a frictionless circular track. P is at the same height as the top of the track, T. (a) Show that the slider cannot reach T. (b) What is the minimum release height for the slider to reach T and carry on round the track?

(a) Energy conservation: P and T are at the same height, so $v_T = 0$ at T. But staying on the track at T needs $mg \leq \frac{mv_T^2}{r}$, i.e. $v_T \geq \sqrt{gr} > 0$. Impossible.
(b) Let the loop have radius r and release height h above the lowest point. Minimum: $v_T^2 = gr$. $mgh = mg(2r) + \frac{1}{2}mv_T^2 \quad \Rightarrow \quad h_{\min} = 2r + \frac{r}{2} = 2.5\,r$